Relativistic MHD in an expanding Universe with Pencil Code

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Alberto Roper Pol

Relativistic magnetohydrodynamics in the early Universe

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The art of simulating the early Universe

A dissertation series about lattice techniques for simulating the dynamics of interacting fields in an expanding Universe

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Part III. Scalar-Gauge-Fluid Dynamics

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Boltzmann equation

$$p^\mu \partial_\mu f_s + m_s F^\mu \partial_{p^\mu} f_s = \mathcal{C}[f_s]$$

First moment -> conservation of mass

$$\partial_\mu J^\mu_m = 0$$

Second moment -> conservation of momentum-energy

$$\partial_\mu T^{\mu\nu} = 0$$

$$J^\mu = m \int p^\mu f \frac{d^3p}{p^0}$$

$$T^{\mu\nu} = m \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

Early Universe fluid (Minkowski space-time)

- Radiation-dominated fluid ($m/T \ll 1$), such that conservation of mass becomes a trivial non-dynamical equation.
- Ultrarelativistic equation of state: $p = \omega\rho = \rho/3$
- Fluid description, assuming small perturbations around local thermal equilibrium

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu + p\eta^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\theta h^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

only justified when $p \ll \rho$

- Consequence of Bianchi identities, $D_\mu G^{\mu\nu} = 0$

$$\theta = \partial_\mu U^\mu\,, \quad h^{\mu\nu} = U^\mu U^\nu + \eta^{\mu\nu}$$

Early Universe fluid (expanding Universe)

- Homogeneous and isotropic Universe described by Friedmann-Lemaître-Robertson-Walker (FLRW) metric tensor

$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu}$$

- Conservation equations do not depend on frame:

$$D_\mu T^{\mu\nu} = 0$$

- Given a relation $g^{\mu\nu} = \Omega^2(x^\mu)\tilde{g}^{\mu\nu}$ between metric tensors we can rewrite

$$\begin{aligned} T^{\mu\nu}{}_{;\mu} &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu\nu}) + \Gamma^\nu_{\mu\sigma}T^{\mu\sigma} \\ &= \frac{\Omega^4}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\,\Omega^{-4}\,T^{\mu\nu}) + \tilde{\Gamma}^\nu_{\mu\sigma}T^{\mu\sigma} - 2\,T^{\mu\nu}\partial_\mu \ln\Omega + T^\mu_{\ \mu}\,\partial^\nu \ln\Omega\,, \end{aligned}$$

Early Universe fluid (expanding Universe)

- Given a relation $g^{\mu\nu} = \Omega^2(x^\mu)\tilde{g}^{\mu\nu}$ between metric tensors we can rewrite

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu\nu}) + \Gamma^\nu_{\mu\sigma}T^{\mu\sigma}$$
$$= \frac{\Omega^4}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\,\Omega^{-4}\,T^{\mu\nu}) + \tilde{\Gamma}^\nu_{\mu\sigma}T^{\mu\sigma} - 2\,T^{\mu\nu}\partial_\mu \ln\Omega + T^\mu_{\ \mu}\,\partial^\nu \ln\Omega\,,$$

- Rescaling $\tilde{T}^{\mu\nu} = \Omega^{-6}T^{\mu\nu}$

$$T^{\mu\nu}{}_{;\mu} = \Omega^6\left(\tilde{T}^{\mu\nu}{}_{;\tilde{\mu}} + \tilde{T}^\sigma{}_\sigma\,\tilde{g}^{\mu\nu}\partial_\mu \ln\Omega\right) = 0 \Leftrightarrow \tilde{T}^{\mu\nu}{}_{;\tilde{\mu}} + \tilde{T}^\sigma{}_\sigma\,\tilde{g}^{\mu\nu}\partial_\mu \ln\Omega = 0$$

$$\tilde{T} = a^4 T = 3\tilde{p} - \tilde{\rho},$$

Conformal invariance

- Since the trace of the stress-energy tensor of a perfect fluid vanishes, the equations in FLRW can be transformed to those in Minkowski space-time under a conformal transformation

$$T^{\mu\nu}{}_{;\,\mu} = 0 \Leftrightarrow \tilde{T}^{\mu\nu}{}_{;\,\tilde{\mu}} = 0\,.$$

$$\tilde{T}^{00} = (\tilde{p} + \tilde{\rho})\gamma^2 - \tilde{p} = \tilde{\rho}_{\rm kin} + \tilde{\rho}$$

$$\tilde{T}^{0i} = (\tilde{p} + \tilde{\rho})\gamma^2 u_i$$

$$\tilde{T}^{ij} = (\tilde{p} + \tilde{\rho})\gamma^2 u_i u_j$$

$$\tilde{\rho}_{\rm kin} = (\tilde{p} + \tilde{\rho})\gamma^2 u^2$$

Coming pressure and energy density:

$$\tilde{p} = a^4 p, \quad \tilde{\rho} = a^4 \rho$$

MHD equations in FLRW

- Then MHD equations in the subrelativistic limit $\gamma^2 \simeq 1$ are (Brandenburg et al., 1996, Roper Pol & Midiri, 2025)

$$\partial_\tau \ln\tilde{\rho} = -\frac{4}{3}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \mathbf{\frac{1}{2}}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}\right] + \frac{1}{\tilde{\rho}}\left[\tilde{\eta}\tilde{\boldsymbol{J}}^2 \mathbf{-}\, \boldsymbol{u}\cdot(\tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}})\right],$$

$$\partial_\tau \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = \frac{\boldsymbol{u}}{3}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \mathbf{\frac{1}{2}}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}\right] - \frac{\boldsymbol{u}}{\tilde{\rho}}\left[\tilde{\eta}\tilde{\boldsymbol{J}}^2 + \mathbf{\frac{1}{2}}\,\boldsymbol{u}\cdot(\tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}})\right] - \frac{1}{4}\boldsymbol{\nabla}\ln\tilde{\rho}$$
$$+ \frac{3}{4\tilde{\rho}}\tilde{\boldsymbol{J}}\times\tilde{\boldsymbol{B}} + \frac{2}{\tilde{\rho}}\,\boldsymbol{\nabla}\cdot(\tilde{\rho}\,\tilde{\nu}\,\tilde{\boldsymbol{\sigma}})\,.$$

- Corrections in red were previously omitted and arise from sub relativistic contributions of $\partial_t \gamma^2$

Generalised fluid equations in FLRW

- The fluid conservation laws can be generalised to any external forces $f^{\nu}_{\rm tot}$ and to relativistic velocities

$$\partial_\tau \ln\tilde{\rho} = -\frac{1+c_{\rm s}^2}{1-c_{\rm s}^2u^2}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1-c_{\rm s}^2}{1-c_{\rm s}^2u^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} + \frac{1}{1-c_{\rm s}^2u^2}\frac{1}{\tilde{\rho}}\left[\tilde{f}^0_{\rm tot}(1+u^2) - 2\,\boldsymbol{u}\cdot\tilde{\boldsymbol{f}}_{\rm tot}\right] + \frac{1-3c_{\rm s}^2}{1-c_{\rm s}^2u^2}(1+u^2)\,\mathcal{H}\,, \tag{1.2a}$$

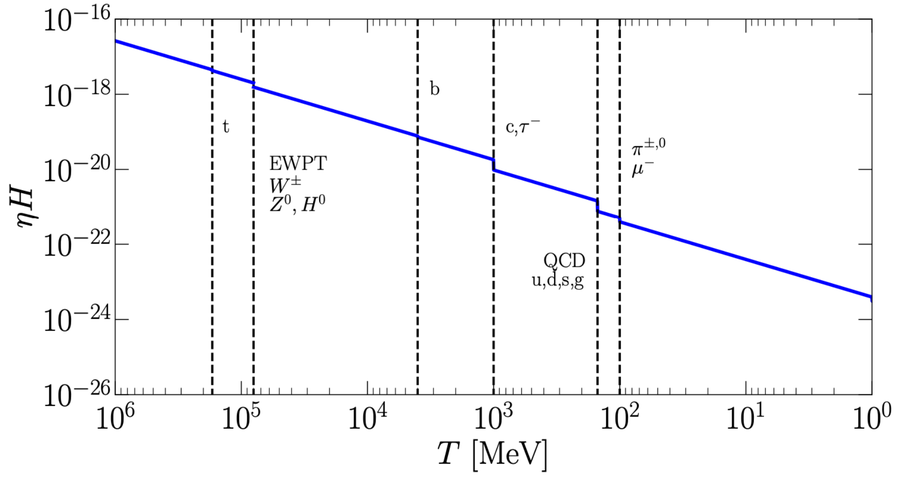
$$D_\tau\boldsymbol{u} = \frac{\boldsymbol{u}}{(1-c_{\rm s}^2u^2)\gamma^2}\left[c_{\rm s}^2\,\boldsymbol{\nabla}\cdot\boldsymbol{u} + c_{\rm s}^2\frac{1-c_{\rm s}^2}{1+c_{\rm s}^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} - \frac{1}{\tilde{\rho}}\left(\tilde{f}^0_{\rm tot} - \frac{2c_{\rm s}^2}{1+c_{\rm s}^2}\boldsymbol{u}\cdot\tilde{\boldsymbol{f}}_{\rm tot}\right) + (3c_{\rm s}^2-1)\,\mathcal{H}\right] - \frac{c_{\rm s}^2}{1+c_{\rm s}^2}\frac{\boldsymbol{\nabla}\ln\tilde{\rho}}{\gamma^2} + \frac{1}{1+c_{\rm s}^2}\frac{\tilde{\boldsymbol{f}}_{\rm tot}}{\tilde{\rho}\gamma^2}\,, \tag{1.2b}$$

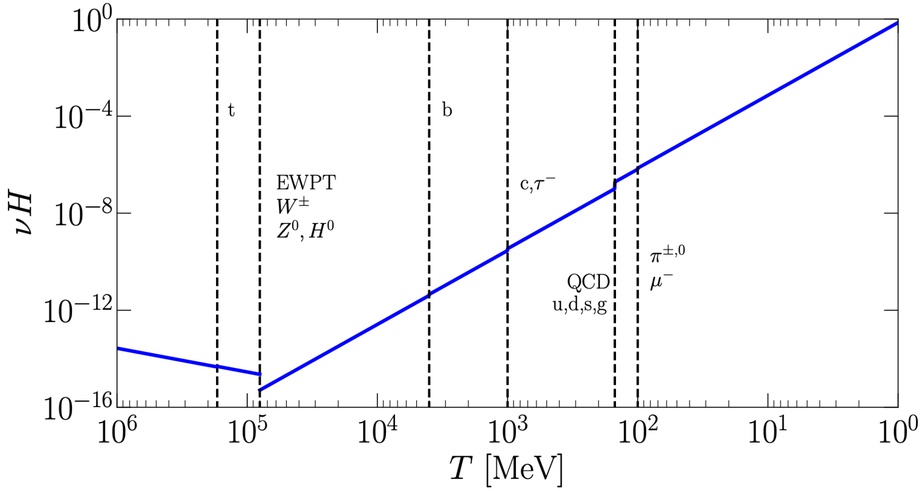
Generalised fluid equations in FLRW

- These expressions can be applied to any external forces, including:
  - Electromagnetic $f^{\nu}_{\rm Lor} \equiv J_{\mu} F^{\nu\mu} = -\partial_{\mu} T^{\mu\nu}_{\rm EM}$
  - Viscous forces $f^{\nu}_{\rm visc} = \partial_{\mu} \Pi^{\mu\nu}$, or for Navier-Stokes,

$$\tilde{f}^{i}_{\rm visc} = \tilde{\nu}\,\partial_j\partial_j u_i + \left(\tfrac{1}{3}\tilde{\nu} + \tilde{\xi}\right)\partial_i\partial_j u_j + \left[2\,\tilde{\nu}\,\partial_{(i}u_{j)} + \left(\tilde{\xi} - \tfrac{2}{3}\tilde{\nu}\right)(\partial_l u_l)\delta_{ij}\right]\partial_j \ln\tilde{\rho}_f$$

  - Interactions with a scalar field $f^{\nu}_{\phi} = \Omega_{\phi}\partial^{\nu}\phi$
  - Reheating $f^{\nu}_{\rm reh} = \Gamma_{\phi}\, T^{\mu\nu}_{\phi}\, u^{\phi}_{\mu}$





Subrelativistic correction in radiation domination

- Energy equation found from temporal component of $\partial_\mu T^{\mu\nu} = 0$

$$D_\tau \ln\tilde{\rho} + D_\tau \ln\gamma^2 + \boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{c_\mathrm{s}^2}{(1+c_\mathrm{s}^2)\gamma^2}\partial_\tau \ln\tilde{\rho}$$

- We can also project the conservation laws along U_ν (relativistic Euler equation)

$$-U_\nu T^{\mu\nu}{}_{;\,\mu} = U^\mu \partial_\mu \rho + (p+\rho)\,\theta = 0 \qquad D_\tau \ln\tilde{\rho} + (1+c_\mathrm{s}^2) D_\tau \ln\gamma + (1+c_\mathrm{s}^2)\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

- Combining both expressions, one directly finds

$$\partial_\tau \ln\tilde{\rho} + \frac{1+c_\mathrm{s}^2}{1-c_\mathrm{s}^2 u^2}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_\mathrm{s}^2}{1-c_\mathrm{s}^2 u^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} = 0 \Rightarrow \lim_{u^2\ll 1} \partial_\tau \ln\tilde{\rho} = -(1+c_\mathrm{s}^2)\boldsymbol{\nabla}\cdot\boldsymbol{u} - (1-c_\mathrm{s}^2)(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}$$

Subrelativistic correction in radiation domination

- This expression is incompatible with directly taking $\gamma^2 \to 1$ in first equation

$$D_\tau \ln\tilde{\rho} + D_\tau \ln\gamma^2 + \boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{c_{\rm s}^2}{(1+c_{\rm s}^2)\gamma^2}\partial_\tau \ln\tilde{\rho} \quad \Rightarrow \partial_\tau\tilde{\rho} + (1+c_s^2)\,\nabla\cdot(\tilde{\rho}\mathbf{u}) = 0$$

- We can again combine the equations to find an equation for γ^2

$$D_\tau \ln\gamma^2 = 2\gamma^2\,\boldsymbol{u}\cdot(D_\tau\boldsymbol{u}) = \frac{2c_{\rm s}^2}{1-c_{\rm s}^2u^2}\left[u^2\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1}{(1+c_{\rm s}^2)\gamma^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}\right]$$

- In the sub relativistic limit it becomes non-negligible!!

$$\lim_{u^2\ll 1} D_\tau \ln\gamma^2 = -\frac{2c_{\rm s}^2}{1+c_{\rm s}^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho}$$

Vorticity generation in a radiation fluid

- Taking the curl of the momentum equation

$$D_\tau \boldsymbol{\omega} + \boldsymbol{\omega}\,\boldsymbol{\nabla}\cdot\boldsymbol{u} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{\boldsymbol{\omega}\Psi}{(1-c_\mathrm{s}^2u^2)\gamma^2} + \boldsymbol{u}\times\boldsymbol{\nabla}\left[\frac{\Psi}{(1-c_\mathrm{s}^2u^2)\gamma^2}\right] + \frac{(\boldsymbol{\nabla}\tilde{p}-\tilde{\boldsymbol{f}}_\mathrm{tot})\times\boldsymbol{\nabla}(\tilde{\rho}\gamma^2)}{(1+c_\mathrm{s}^2)\tilde{\rho}^2\gamma^4} + \frac{\boldsymbol{\nabla}\times\tilde{\boldsymbol{f}}_\mathrm{tot}}{(1+c_\mathrm{s}^2)\tilde{\rho}\gamma^2}$$

$$\Psi = c_\mathrm{s}^2\boldsymbol{\nabla}\cdot\boldsymbol{u} + c_\mathrm{s}^2\frac{1-c_\mathrm{s}^2}{1+c_\mathrm{s}^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} - \frac{\tilde{f}^0_\mathrm{tot}}{\tilde{\rho}} + \frac{2\,c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\frac{\boldsymbol{u}\cdot\tilde{\boldsymbol{f}}_\mathrm{tot}}{\tilde{\rho}} + (3c_\mathrm{s}^2-1)\mathcal{H}$$

- Even in the absence of external forces and baroclinic term, vorticity can be produced from misalignments between $\mathbf{u}$ and gradients of Ψ or u^2

Conservation form of fluid equations

- The conservation laws can be directly written in their conservation form, for any external forces

$$\partial_\mu \tilde{T}^{\mu\nu} = \tilde{f}^\nu \Rightarrow \quad \begin{aligned} &\partial_\tau \tilde{T}^{00} + \partial_i \tilde{T}^{0i} = \tilde{f}^0 \\ &\partial_\tau \tilde{T}^{0i} + \partial_j \tilde{T}^{ij} = \tilde{f}^i \end{aligned}$$

- Requires to construct $\tilde{T}^{ij} = (\tilde{p} + \tilde{\rho})\gamma^2 u_i u_j$ in terms of the dynamical variables $\tilde{T}^{00} = (\tilde{p} + \tilde{\rho})\gamma^2 - \tilde{p}$ and $\tilde{T}^{0i} = (\tilde{p} + \tilde{\rho})\gamma^2 u_i$

Conservation form of fluid equations

- We then define $z \equiv \tilde{T}^{00}/\tilde{\rho} = (1+\omega)\gamma^2 - \omega$ where $\omega = \tilde{p}/\tilde{\rho}$ (e.g. $\omega = 1/3$)
- The following relation holds

$$r^2 = \left[\frac{T^{0i}_{\rm pf}}{T^{00}_{\rm pf}}\right]^2 = \frac{(z+\omega)(z+1)}{z^2}$$

which we can invert to compute z

$$z = \frac{1}{2(1-r^2)}\left(1-\omega+\sqrt{(1+\omega)^2-4\omega r^2}\right)$$

- We can then reconstruct the fluid primitive variables and $\tilde{T}^{ij}$, as required to advance the dynamical equations

$$u_i = \frac{z}{z+\omega}\frac{T^{0i}_{\rm pf}}{T^{00}_{\rm pf}}, \quad \rho_f = \frac{1}{z}T^{00}_{\rm pf} \quad \gamma^2 = \frac{z+\omega}{1+\omega} \quad T^{ij}_{\rm pf} = w_f\gamma^2 u_i u_j + p\delta_{ij} = \frac{z}{1+\omega}\frac{T^{0i}_{\rm pf}T^{0i}_{\rm pf}}{[T^{00}_{\rm pf}]^2} + \frac{\omega}{z}T^{00}_{\rm pf}\delta_{ij}$$

Conservation form of fluid equations

- Note that in the subrelativistic limit we can not just take the limit $\gamma^2 \to 1$ to relate the components in the conservation form

$$\tilde{T}^{00} = \tilde{\rho}\,, \qquad \tilde{T}^{0i} = (1+c_{\rm s}^2)\,\tilde{\rho} u^i\,, \qquad \tilde{T}^{ij} = \frac{1}{1+c_{\rm s}^2}\frac{\tilde{T}^{0i}\tilde{T}^{0j}}{\tilde{T}^{00}} + c_{\rm s}^2\,\tilde{T}^{00}\,\delta^{ij}\,.$$

- This would be equivalent to the incomplete equations we have seen before in the non-conservation form! We would be neglecting contributions from $\partial_\tau \gamma^2$
- We need to expand γ^2 up to second order in r^2 and keep next-to-leading order term

$$\lim_{|\vec{u}_{\rm p}|^2 \ll 1} \tilde{T}^{ij}_{\rm pf} = \frac{1}{1+\omega}\frac{\tilde{T}^{0i}_{\rm pf}\tilde{T}^{0j}_{\rm pf}}{\tilde{T}^{00}_{\rm pf}}\left(1+\frac{r^2\omega}{(1+\omega)^2}\right) + \omega\,\tilde{T}^{00}_{\rm pf}\left(1-\frac{r^2}{1+\omega}\right)\delta_{ij}$$

Electromagnetic fields

- Covariant electromagnetic fields

$$E_\mu = F_{\mu\nu}\, U^\nu\,, \qquad B^\mu = \mathcal{F}^{\mu\nu}\, U_\nu\,, \qquad F_{\mu\nu} = \mathcal{E}_{\mu\nu\alpha\beta} B^\alpha U^\beta + U_\mu E_\nu - U_\nu E_\mu$$

- Coming magnetic fields

$$\tilde{E}^\mu \equiv a^3 E^\mu = \gamma\left(\tilde{\boldsymbol{E}}\cdot\boldsymbol{u}, \tilde{\boldsymbol{E}} + \boldsymbol{u}\times\tilde{\boldsymbol{B}}\right), \qquad \tilde{B}^\mu \equiv a^3 B^\mu = \gamma\left(\tilde{\boldsymbol{B}}\cdot\boldsymbol{u}, \tilde{\boldsymbol{B}} - \boldsymbol{u}\times\tilde{\boldsymbol{E}}\right),$$

- Careful! Not to confuse with electromagnetic fields boosted into the frame of the observer with four-velocity $U^\mu = \gamma(1, u_i)$

$$\begin{aligned}\boldsymbol{E}' &= \gamma\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) - (\gamma-1)(\boldsymbol{E}\cdot\hat{\boldsymbol{u}})\,\hat{\boldsymbol{u}}\,,\\ \boldsymbol{B}' &= \gamma\left(\boldsymbol{B} - \boldsymbol{u}\times\boldsymbol{E}\right) - (\gamma-1)(\boldsymbol{B}\cdot\hat{\boldsymbol{u}})\,\hat{\boldsymbol{u}}\,,\end{aligned}$$

Maxwell equations

- Maxwell equations in curved space-time

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = J^\mu$$

- Directly seen that are conformal invariant (radiation component)

$$\partial_\nu \tilde{F}^{\mu\nu} = \tilde{J}^\mu$$

- Ohm's law

$$J^\mu = \rho_e\, U^\mu + \sigma\, E^\mu$$

$$\tilde{J}^0 = \gamma(\tilde{\rho}_e + \tilde{\sigma}\,\boldsymbol{u}\cdot\tilde{\boldsymbol{E}})\,, \qquad \tilde{J}^i = \gamma(\tilde{\rho}_e\boldsymbol{u} + \tilde{\sigma}[\tilde{\boldsymbol{E}} + \boldsymbol{u}\times\tilde{\boldsymbol{B}}])\,.$$

$$\tilde{J}^0 = \tilde{\rho}_e/\gamma + \boldsymbol{u}\cdot\tilde{\boldsymbol{J}}$$

$$\tilde{\boldsymbol{E}} = \frac{\tilde{\eta}}{\gamma}\,\tilde{\boldsymbol{J}} - \tilde{\eta}\,\tilde{\rho}_e\,\boldsymbol{u} - \boldsymbol{u}\times\tilde{\boldsymbol{B}}$$

- Charge density and conductivity are Lorentz scalars

$$\rho_e = -J_\mu U^\mu\,, \qquad \sigma\, E^\mu = J_\nu h^{\mu\nu}\,.$$

Maxwell equations

- Maxwell equations for an expanding Universe

$$\partial_\tau \tilde{\boldsymbol{E}} = \boldsymbol{\nabla} \times \tilde{\boldsymbol{B}} - \tilde{\boldsymbol{J}}\,, \qquad \partial_\tau \tilde{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}\,, \qquad \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{E}} = \tilde{J}^0\,, \qquad \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{B}} = 0\,.$$

- In MHD, we neglect the displacement current for large conductivity, we get induction equation

$$\partial_\tau \tilde{\boldsymbol{B}} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \tilde{\boldsymbol{B}}) + \frac{\tilde{\eta}}{\gamma} \boldsymbol{\nabla}^2 \tilde{\boldsymbol{B}} - \boldsymbol{\nabla} \times [\tilde{\boldsymbol{B}} \times \boldsymbol{\nabla}(\tilde{\eta}/\gamma)] + \boldsymbol{\nabla} \times (\tilde{\eta}\, \tilde{\rho}_e \boldsymbol{u})$$

$$D_\tau \boldsymbol{A} = \boldsymbol{u} \cdot (\boldsymbol{\nabla} \boldsymbol{A}) + \frac{\tilde{\eta}}{\gamma} (\boldsymbol{\nabla}^2 \boldsymbol{A} - \boldsymbol{\nabla} \Gamma) + \tilde{\eta}\, \tilde{\rho}_e \boldsymbol{u}\,. \qquad \Gamma = \nabla \cdot \mathbf{A}$$

Conservation and non-conservation forms of MHD equations

$$\partial_\mu \tilde{T}^{\mu\nu} = \tilde{f}^{\nu}_{\rm Lor} \Rightarrow \partial_\tau \tilde{T}^{00} + \partial_i \tilde{T}^{0i} = \tilde{f}^{0}_{\rm Lor}$$

$$\partial_\tau \tilde{T}^{0i} + \partial_j \tilde{T}^{ij} = \tilde{f}^{i}_{\rm Lor}$$

$$\begin{aligned}\partial_\tau \ln\tilde{\rho} =& -\frac{1+c_{\rm s}^2}{1-c_{\rm s}^2u^2}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \frac{1-c_{\rm s}^2}{1-c_{\rm s}^2u^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} \\ &+ \frac{1}{1-c_{\rm s}^2u^2}\frac{1}{\tilde{\rho}}\left[(\tilde{f}^0_{\rm ipf}+\tilde{f}^0_{\rm Lor})(1+u^2) - 2\,\boldsymbol{u}\cdot(\tilde{\boldsymbol{f}}_{\rm ipf}+\tilde{\boldsymbol{f}}_{\rm Lor})\right] + \mathcal{F}^0_H\,, \qquad (6.31a)\end{aligned}$$

$$\begin{aligned}D_\tau\boldsymbol{u} =& \frac{\boldsymbol{u}}{(1-c_{\rm s}^2u^2)\gamma^2}\left[c_{\rm s}^2\,\boldsymbol{\nabla}\cdot\boldsymbol{u} + c_{\rm s}^2\frac{1-c_{\rm s}^2}{1+c_{\rm s}^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde{\rho} - \frac{1}{\tilde{\rho}}\left(\tilde{f}^0_{\rm ipf}+\tilde{f}^0_{\rm Lor} - \frac{2c_{\rm s}^2}{1+c_{\rm s}^2}\boldsymbol{u}\cdot[\tilde{\boldsymbol{f}}_{\rm ipf}+\tilde{\boldsymbol{f}}_{\rm Lor}]\right)\right] \\ &- \frac{c_{\rm s}^2}{1+c_{\rm s}^2}\frac{\boldsymbol{\nabla}\ln\tilde{\rho}}{\gamma^2} + \frac{1}{1+c_{\rm s}^2}\frac{\tilde{\boldsymbol{f}}_{\rm ipf}+\tilde{\boldsymbol{f}}_{\rm Lor}}{\tilde{\rho}\gamma^2} + \boldsymbol{\mathcal{F}}_H\,, \qquad (6.31b)\end{aligned}$$

$$D_\tau\boldsymbol{A} = \boldsymbol{\nabla}\chi + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{A}) + \frac{\tilde{\eta}}{\gamma}\left[\boldsymbol{\nabla}^2\boldsymbol{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})\right] + \tilde{\eta}\,\tilde{\rho}_e\boldsymbol{u} \qquad \mathbf{J} = \nabla\times\mathbf{B}$$

Bonus track: Boris correction