Hands-on Session #1b “MHD waves: Alfvén damped waves”

1 Theoretical background

In this exercise, first we will derive a system of linear waves that are obtained in the plasma according to MHD equations and then we will reproduce numerically and study a particularly relevant type of MHD waves: Alfvén waves.

1.1 System of equations

For this exercise, we are going to consider the system of MHD equations with

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0\,, \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u}\,, \tag{2}$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}\,. \tag{3}$$

For the pressure equation, we will consider the adiabatic equation of state

$$\mathrm{d}(p\rho^{-\gamma})/\mathrm{d}t = 0\,, \tag{4}$$

with γ being the adiabatic exponent, ratio of the specific heats $\gamma = c_p/c_v$, equal to 5/3 for a simple gas. This is equivalent to an isentropic flow $\mathrm{d}s/\mathrm{d}t = 0$ with $s = c_v \ln(p\rho^{-\gamma})$.

Exercise 1 *Derive the equation for the pressure evolution taking into account that* $\mathrm{d}/\mathrm{d}t$ *represents a total derivative* $\rightarrow \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$

Exercise 2 *Consider perturbations of the type* $\tilde{v} = v_1 \exp(i\boldsymbol{k} \cdot x - i\omega t)$ *and similar for* p_1, ρ_1, b_1, *over a homogeneous background described by* $\boldsymbol{u}_0 = 0$, p_0, ρ_0, *and* $\boldsymbol{B}_0$. *Hint: You will need to include Maxwell equations to relate* $\boldsymbol{J}$ *and* $\boldsymbol{B}$ *under the common assumption of negligible displacement current in MHD.*

Exercise 3 *Taking the coordinate system* $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$, $\boldsymbol{k} = k_\perp \boldsymbol{e}_y + k_\parallel \boldsymbol{e}_z$, *show that the system of perturbations can be found to be (after solving explicitly for* $\boldsymbol{v}_1$ *and neglecting viscous and diffusive effects),*

$$\begin{pmatrix} \omega^2 - k_\parallel^2 v_\mathrm{A}^2 & 0 & \\ 0 & \omega^2 - k_\perp^2 c_\mathrm{s}^2 - k^2 v_\mathrm{A}^2 & -k_\perp k_\parallel c_\mathrm{s}^2 \\ 0 & -k_\perp k_\parallel c_\mathrm{s}^2 & \omega^2 - k_\parallel^2 c_\mathrm{s}^2 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

The Alfvén speed is $v_\mathrm{A} = B_0/\sqrt{\rho_0}$ and the sound speed is $c_\mathrm{s} = \sqrt{\gamma p_0/\rho_0}$. Note that $k^2 = k_\perp^2 + k_\parallel^2$.
The eigenvalues of this equation are given for a vanishing determinant,

$$(\omega^2 - k_\parallel^2 v_\mathrm{A}^2)\big(\omega^4 - \omega^2 k^2 (c_\mathrm{s}^2 + v_\mathrm{A}^2) + k^2 c_\mathrm{s}^2 k_\parallel^2 v_\mathrm{A}^2\big) = 0\,. \tag{6}$$

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Exercise 4 *Find the solutions to the previous equation. You should find shear Alfvén waves, compressional Alfvén waves, also called fast magnetosonic waves, and slow magnetosonic waves.*

For the numerical exercise, we will focus on shear Alfvén waves, which have dispersion relation $\omega^2 = k_\parallel^2 v_{\rm A}^2$ and play an important role in MHD and MHD turbulence. In this mode, the fluid perturbations are incompressible and perpendicular to the magnetic field, $v_1 = (v_x, 0, 0)$, with $\boldsymbol{k} \cdot \boldsymbol{v}_1 = 0$. The magnetic field perturbations $\boldsymbol{b}_1$ are perpendicular to $\boldsymbol{B}_0$ and have amplitude $b_1 = \pm\sqrt{\rho_0} v_1$, which originates from the $\boldsymbol{B} \cdot \nabla \boldsymbol{B}$ term in the Lorentz force, as ∇B^2 becomes of second order.

Exercise 4 *Focusing only on this mode, perturb the original MHD equations to find the Alfvén wave modes without excluding viscous and diffusive terms. This is the system that we will simulate on* PENCIL CODE.

2 Code Setup

The setup for this exercise is a little bit special and will allow you to see the flexibility and modularity of the Pencil Code to add alternative functions and solvers.
You can find the sample to be run under samples/damped_alfven_waves.
The physics modules that are used in this setup are; see src/Makefile.local:

```
MPICOMM    =   mpicomm

DENSITY    = nodensity
HYDRO      =   hydro
MAGNETIC   =   bfield

INITIAL_CONDITION   =   initial_condition/alfven_wave
IMPLICIT_DIFFUSION =   implicit_diffusion
FOURIER             =   fourier_fftpack

REAL_PRECISION = double
```

This means that by default this sample is set to be run in parallel (using 'mpicomm'). The first thing you might want to do is to change this to 'nompicomm', such that you run the code without using MPI (message passing interface) and then modify src/cparam.local to run on only one processor (also make sure you choose one processor per direction). You don't need to change the number of grid points (can/should be kept at 32)!

The "hydro" and "magnetic" are selected such that we solve for the momentum and the induction equations. However, the 'bfield' option is chosen for 'magnetic'. This means that we are directly solving for the magnetic field $\boldsymbol{B}$ instead of for the vector potential $\boldsymbol{A}$, as commonly done in PENCIL CODE.

You can check the initial conditions under src/initial_condition/alfven_wave.f90.

Exercise 5 *Look into the initial condition/Alfven wave Fortran file and write down which are the initial conditions for the velocity and magnetic field being used for this run.*

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Exercise 6 *What are the values of the viscosity and magnetic diffusivity being run in the code?*

Exercise 7 *Write down the system of equations that are being solved for this sample*

Exercise 8 *Run the code and plot the time evolution of the magnetic field components in the parallel and perpendicular directions to the background. Do the same for the velocity field. Check that your run in series gives the same result as the reference in the sample reference.out*

Exercise 9 *Generate a contour plot with the value of the velocity field in the yz-plane. Can you see what is the direction of propagation? What is the speed of propagation? Are they consistent with the input parameters of the simulation?*

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