Hands-on Session #1a Nonlinear sounds waves

1 Theoretical background

1.1 System of equations

Most problems in incompressible flow involve only two unknowns: pressure and velocity, which are typically found by solving the two equations that describe the conservation of mass and of linear momentum, with the fluid density presumed constant.
In compressible flow, however, the gas density and temperature also become variables. This requires two more equations in order to solve compressible-flow problems: an equation of state for the gas and a conservation of energy equation.
The system of equations for a compressible flow is

$$\frac{\partial \rho}{\partial t} + \nabla\left(\rho \boldsymbol{v}\right) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \frac{1}{m}\boldsymbol{F} + \frac{\mu}{\rho}\left(\nabla^2\boldsymbol{v} + \frac{1}{3}\nabla\left(\nabla\cdot\boldsymbol{v}\right)\right), \tag{2}$$

$$\rho\left(\frac{\partial \epsilon}{\partial t} + \boldsymbol{v}\cdot\nabla\epsilon\right) = \nabla\cdot\left(K\nabla T\right) - p\nabla\cdot\boldsymbol{v} \tag{3}$$

plus an equation of state. For the majority of gas-dynamic problems, the ideal gas law is the appropriate state equation.

1.2 Qualitative description

We consider perturbations of gas, where self-gravity is negligible (like air). The resulting access pressure that tries to smooth out the compression, which leads to acoustic waves.

1.3 Quantitative description

Introduce unperturbed and perturbed parts:

- density: $\rho \to \rho_0 + \rho_1(\boldsymbol{x}, t)$
- pressure: $p \to p_0 + p_1(\boldsymbol{x}, t)$
- velocity: $\boldsymbol{v} \to \boldsymbol{v}_1(\boldsymbol{x}, t)$ (assume no mean flow, $\boldsymbol{v}_0 = 0$)

Further assumptions in the analytical analysis:

- Mean fields do not depend on space and time.
- Perturbations are small ($\rho_1 \ll \rho_0$, $p_1 \ll p_0$) [This means we can keep only linear terms the *principle of superposition* holds: any perturbation can be treated as a superposition of many Fourier modes and each mode can be studied in isolation.]

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- Ideal fluids (no transport phenomena), but compressible.
- No external forces ($\boldsymbol{F} = 0$).
- Assume that the viscosity is negligible.

By inserting the decomposition in perturbed and unperturbed parts into Eqns. (2) and (3), and by introducing the sound speed

$$c_\mathrm{s} = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} \approx \sqrt{\frac{p_1}{\rho_1}}, \tag{4}$$

one finds that

$$\left(\frac{\partial^2}{\partial t^2} - c_\mathrm{s}^2 \nabla^2\right) \rho_1 = 0 \tag{5}$$

Note:

- Eq. (5) is valid for any fluid, the detailed properties of the gas only enter when the sound speed is evaluated.
- In very incompressible fluids, $\mathrm{d}\rho$ changes only little with $\mathrm{d}p$, therefore the sound speed is large. (Indeed, $c_\mathrm{s} = 332\ \mathrm{m\,s^{-1}}$ for air while $c_\mathrm{s} = 1430\ \mathrm{m\,s^{-1}}$ or very incompressible water.) For a truly incompressible fluid $c_\mathrm{s} = \infty$.

Since our analysis is linear, the principle of superposition holds and we can substitute a Fourier component with wavenumber $\boldsymbol{k}$

$$\rho_1 = \rho_{1,0} \exp\left(i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right) \tag{6}$$

into (5) and find the dispersion relation:

$$\boxed{\omega^2 = c_\mathrm{s}^2 k^2}. \tag{7}$$

Note:

- Both group and phase velocity, $v_\mathrm{p} = \omega/k$ and $\boldsymbol{v}_\mathrm{g} = \nabla_{\boldsymbol{k}}\omega$, are equal to c_s for sound waves.
- Since the speed does not depend on the frequency, sound waves are called *non-dispersive*. But this can be different in stratified gas.
- Sound waves are always *longitudinal*.

2 Code setup

The simulation setup NonlinearSound is designed to explore the propagation of acoustic waves. The physics modules that are used in this setup are; see src/Makefile.local:

```
HYDRO     =   hydro
DENSITY   =   density
ENTROPY   =   entropy
MAGNETIC  = nomagnetic
GRAVITY   = nogravity
```


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This means we are solving the momentum equation ("hydro"), the continuity equation ("density"), and the entropy equation ("entropy"). The reason for the implementation of the entropy in the code rather than the thermal energy, is related to Solar physics. There, entropy is the natural physical variable for (at least) convection processes: the sign of the entropy gradient determines convective (in)stability, the Rayleigh number is proportional to the entropy gradient of the associated hydrostatic reference solution, etc.

The grid size and number of CPUs is set in src/cparam.local:

```
integer, parameter :: ncpus=1,nprocx=1,nprocy=1,nprocz=1
integer, parameter :: nxgrid=128,nygrid=1,nzgrid=1
```

We use one core ("ncpus=1"), and a 1D grid with 128 meshpoints ("nxgrid=128").

The most relevant setting for the initial condition, see start.in, are

```
&hydro_init_pars
inituu='sound-wave', ampluu=1e-3
/
&density_init_pars
initlnrho='sound-wave', ampllnrho=1e-3
/
&entropy_init_pars
/
```

This corresponds to $\boldsymbol{u} = (u_x, 0, 0)$ with $u_x = A_{\rm u} c_{\rm s,0} \sin(kx)$ and $\ln\rho = A_\rho \sin(kx)$, where $c_{\rm s,0} = 1$ by default, and $A_{\rm u} = A_\rho = 10^{-3}$ is chosen.

The most relevant run parameter is the viscosity which, in the provided setup, is chosen to be very small; see run.in:

```
&viscosity_run_pars
nu=1e-5
/
```

The default output, as set in print.in, includes:

- it = number of time step
- t = time
- dt = time step
- urms = $\langle \boldsymbol{u}^2 \rangle^{1/2}$ = rms velocity
- ekin = kinetic energy
- ethm = thermal energy
- epsK = rate of kinetic energy dissipation
- ugradpm = $\langle \boldsymbol{u} \cdot \nabla p \rangle$ = the work done by the pressure gradient
- rhom = $\langle \rho \rangle$ = mean density = mass/volume
- ssm = $\langle s \rangle$ = mean entropy.

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3 Exercises

(a) Verify the derivation in linear perturbation theory of the dispersion relation (7).

(b) Now, test your result with a simulation. First, link the source files

```
> pc_setupsrc
```

and then compile the code by

```
> pc_build
```

Initialize the simulation by running

```
> pc_start
```

Check where and how the initial conditions are programmed in the code. Search for initlnrho='sound-wave' using the "grep" command. Then plot lnrho vs. x for the VAR0 to see if it has the properties that you expect.

(c) Now, run

```
> pc_run
```

to obtain the time evolution. Analyze the time evolution data and the snapshots, following the Jupyter notebook instructions. What happens as time goes on? Does the wave travel? If so, in which direction?

(d) Determine empirically the speed with which the initial sine wave propagates.

(e) Modify the initial condition such that the wave propagates in the opposite direction.

(f) Repeat the experiment with $\nu = 0.05$. Does the speed of the wave change? Does the amplitude change? Compare with theoretical expectation.

(g) Change the amplitude to unity, i.e., put $A_{\rm u} = A_\rho = 1$. Describe what happens. By how much does the kinetic energy decrease? What happens to the thermal energy?

(h) Can you also get shocks with with $A_{\rm u} = A_\rho = 10^{-3}$?

(i) For $A_{\rm u} = A_\rho = 1$, how much do you need to increase the viscosity to avoid wiggles?

(j) Change the order of the scheme (itorder=2 or 1), to find out the error in energy conservation.

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