1st Pencil Code school on early Universe physics and gravitational waves

CERN, 20-24 October 2025

Magnetohydrodynamics in the early Universe

Lecturer: ***Alberto Roper Pol***
(University of Geneva)

Accelerating the Pencil Code

CERN

Pencil Code User Meeting
October 2025 - CERN (Switzerland)

Teaching material

- Lecture notes from EPFL course “Numerical simulations of magnetohydrodynamics in the early Universe” covering:
    - ***Basics of fluid dynamics***
    - ***Magnetohydrodynamics***
    - ***Turbulence***
    - ***Early Universe MHD*** (intro to cosmology, Maxwell equations, axion electrodynamics and inflation, MHD during radiation-domination era, chiral anomaly, gravitational waves)

  https://jennifer-schober.com/ [password: BernoulliMHD2024]

- Review published on arxiv with updated results for MHD in an expanding Universe: https://arxiv.org/pdf/2501.05732

Numerical simulations of magnetohydrodynamics in the early Universe

Alberto Roper Pol and Jennifer Schober

Spring Semester 2024

If you find typos, please let us know!

arXiv:2501.05732v2 [gr-qc] 3 Jul 2025

Relativistic magnetohydrodynamics in the early Universe

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Abstract. We review the conservation laws of magnetohydrodynamics (MHD) in an expanding homogeneous and isotropic Universe that can be applied to the study of early Universe physics during the epoch of radiation domination. The conservation laws for a conducting perfect fluid with relativistic bulk velocities in an expanding background are presented, extending previous results that apply in the limit of subrelativistic bulk motion. Furthermore, it is shown that the subrelativistic limit presents new corrections that have not been considered in previous work. We discuss the conformal invariance of the MHD equations for a radiation-dominated fluid and different types of scaling of the fluid variables that are relevant for other equations of state when the bulk velocity is subrelativistic. In particular, we review the super-comoving coordinates that scale the time coordinate with the Universe expansion, and present a particular choice that allows the equations to become conformally flat for any choice of the equation of state. Imperfect relativistic fluids are briefly described but their detailed study is not included in this work. We review the propagation of sound waves, Alfvén waves, and magnetosonic waves in the early Universe plasma. The Boris correction for relativistic Alfvén speeds is presented and adapted for early Universe applications. This review is an extension, including new results, of part of the lectures presented at the minicourse “Simulations of Early Universe Magnetohydrodynamics” lectured by A. Roper Pol and J. Schober at EPFL, as part of the six-week program “Generation, evolution, and observations of cosmological magnetic fields” at the Bernoulli Center in May 2024.

Keywords: magnetohydrodynamics, early universe, cosmology

Submitted to: *Rep. Prog. Phys.*

Recommended literature

General (magneto-)hydrodynamics:

Rai Choudhuri, "*The physics of fluids and plasmas. An introduction for astro-physicists*", Cambridge University Press, 1998 [Link]

Biskamp, "*Magnetohydrodynamic Turbulence*", Cambridge University Press, 2003 [Link]

Davidson, "Turbulence : an introduction for scientists and engineers" Oxford University Press, 2004 [Link]

Rogachevskii, "Introduction to Turbulent Transport of Particles, Temperature and Magnetic Fields", Cambridge University Press, 2021 [Link]

Rezzolla & Zanotti, "Relativistic Hydrodynamics", Oxford University Press, 2013 [Link]

Astrophysical magnetic fields:

Shukurov & Subramanian, "Astrophysical Magnetic Fields From Galaxies to the Early Universe", Cambridge University Press, 2021 [Link]

Brandenburg & Subramanian, "Astrophysical magnetic fields and nonlinear dynamo theory", Physics Reports, 2005 [Link]

General cosmology and gravitational waves:

Baumann, "Cosmology", Cambridge University Press, 2022 [Link]

Liddle, "An Introduction to Modern Cosmology", Wiley-VCH , 2003 [Link]

Maggiore, "Gravitational Waves", Oxford University Press, 2018 [Link]

Early Universe magnetohydrodynamics:

Durrer & Neronov, "Cosmological magnetic fields: their generation, evolution and observation", The Astronomy and Astrophysics Review, 2013 [Link]

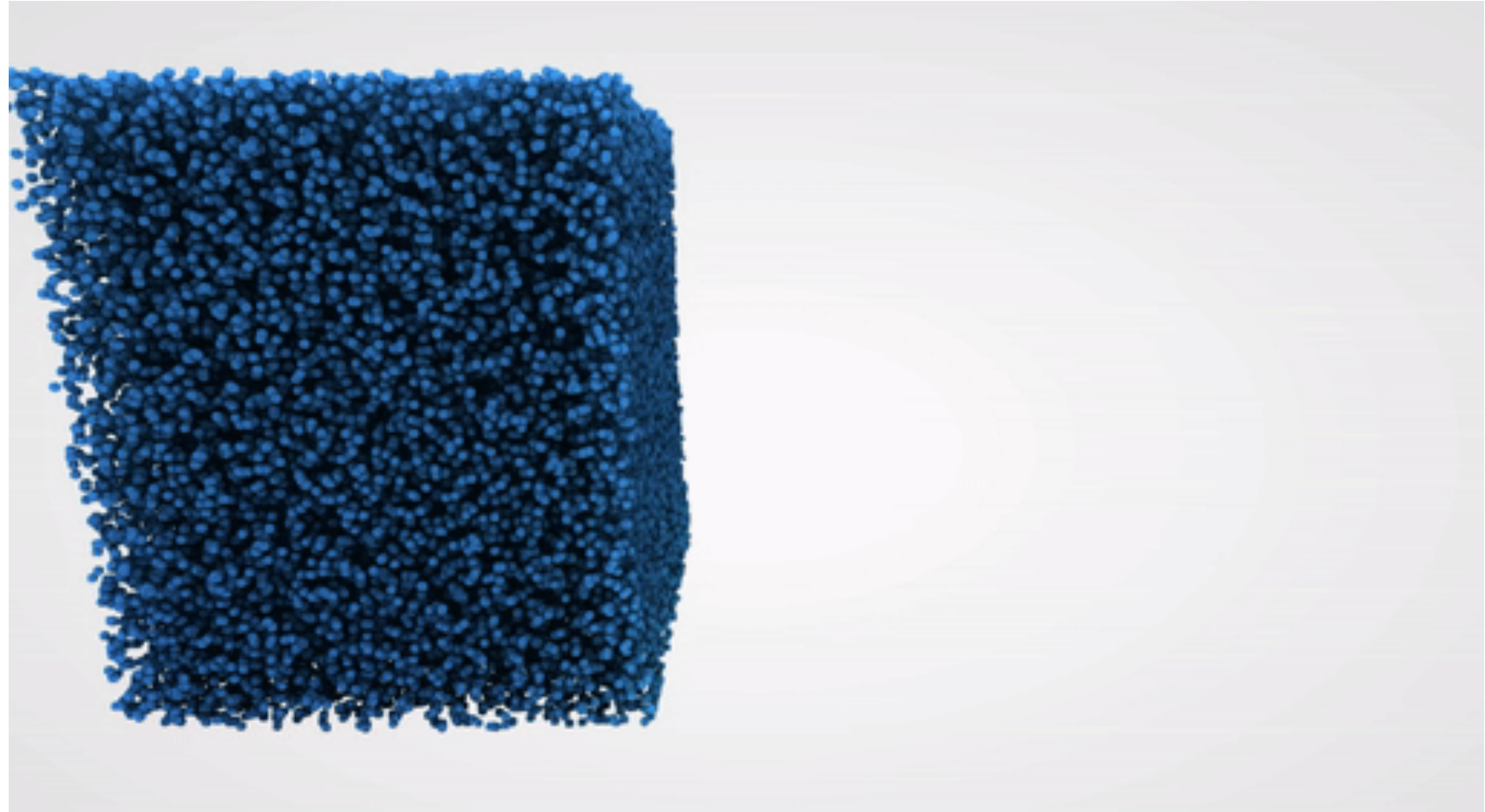
Subramanian, "The origin, evolution and signatures of primordial magnetic fields", Reports on Progress in Physics, 2016 [Link]

Vachaspati, "Progress on cosmological magnetic fields", Reports on Progress in Physics, 2021 [Link]

- **Fluid:** ensemble of particles that presents collective motion and deforms (flows) under shear stresses

- **Plasma:** electrically quasi-neutral medium of charged particles that interact collectively

Lagoon Nebula is a large, low-density cloud of partially ionized gas.[1]
From wikipedia



https://experiments.withgoogle.com/fluid-particles

Different levels of study of plasmas

- **Level 0:** ensemble of N quantum particles (length scales around and below the de Broglie wave length of the particles)

- **Level 1:** ensemble of N classical particles (length scales much larger than the de Broglie wave length), we can ignore quantum effects and treat the system classically -> **kinetic theory**

$$R = \lambda_{\mathrm{DB}}/l \ll 1$$

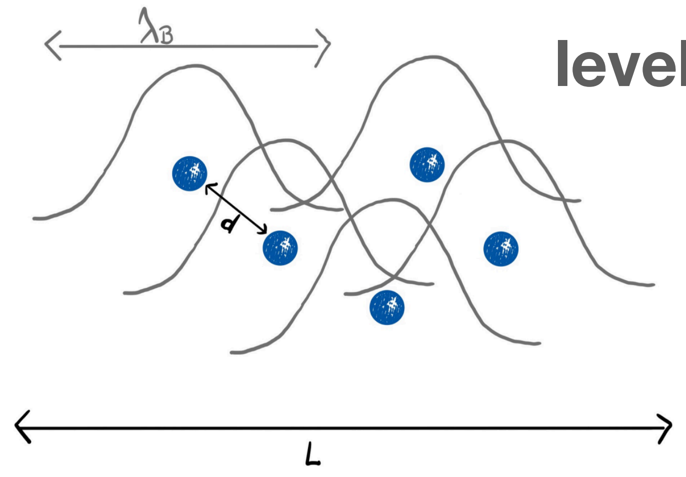
- **Level 2:** distribution function f_s can be used to represent local properties of a fluid s (ensemble of particles of species s). It corresponds to the distribution of particles per unit of 6-dimensional volume (space and momentum) -> **Boltzmann** (neutral collisional fluids) or **Vlasov** (collisionless plasmas) **equation**

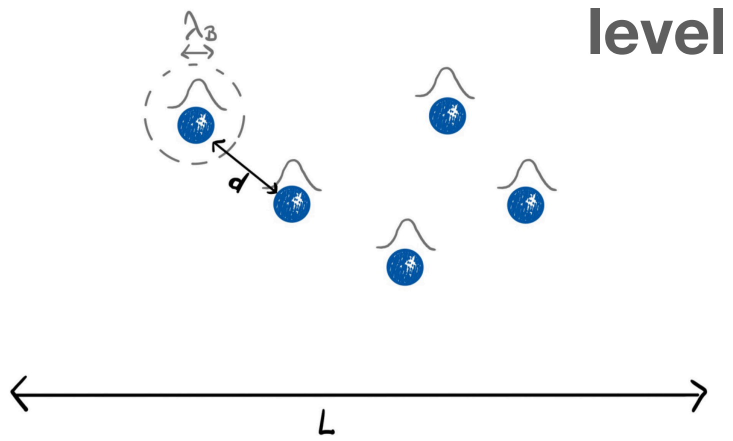
$$N_s = \int f_s(t, x, v)\ d^3x\ d^3v$$

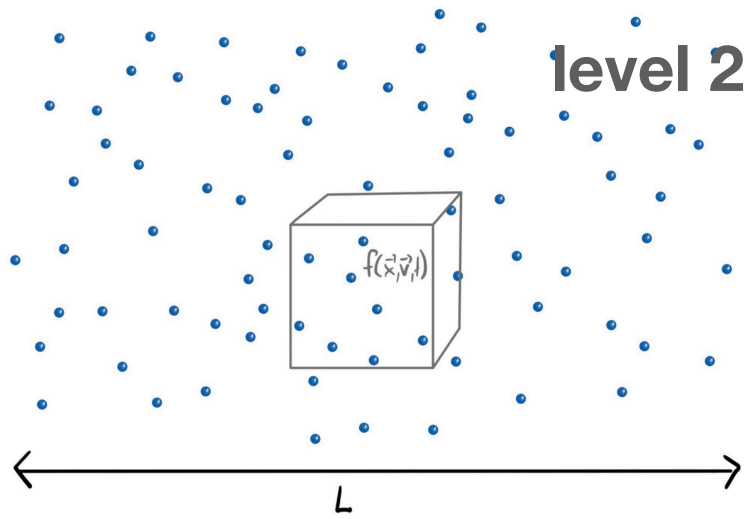
- **Level 3: multi-fluid equations** (hydrodynamics for single fluid) are obtained as a hierarchical system of moments of the Boltzmann equation up to any order. Infinite system of equations require a truncation (hierarchy problem). We usually take an equation of state to close the system and assume that the length scales of the fluid are much larger than the mean-free path (small Knudsen number)

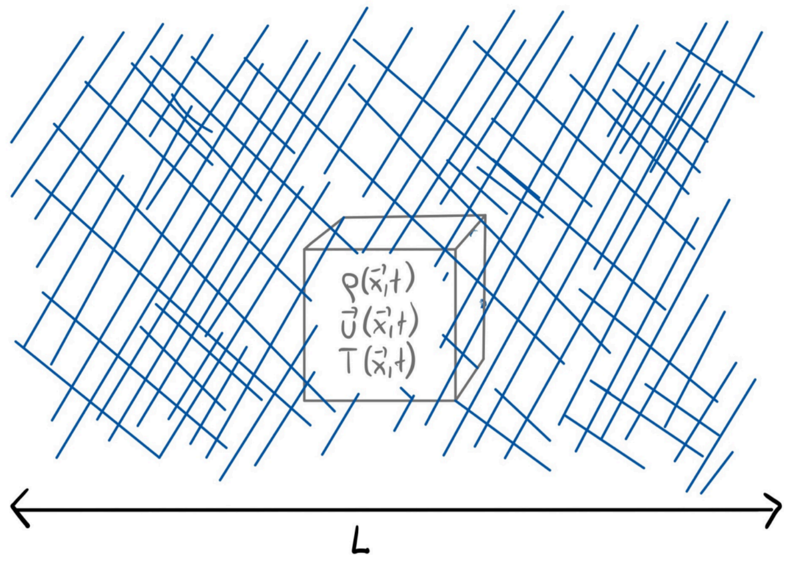
$$\mathrm{Kn} = l_{\mathrm{mfp}}/l \ll 1$$

- **Level 4: Magnetohydrodynamics** is the study of a plasma obtained after combining the fluid equations for charged particles (e.g. ions and electrons) and treat the system as a single fluid, coupled to Maxwell equations, usual assumptions are neutrality and large conductivity (neglecting the displacement current)


level 0


level 1


level 2


levels 3 and 4

Credit: J. Schober

Kinetic theory of plasmas

Klimontovich equation (discrete ensemble of particles)

$$N(\boldsymbol{x},\boldsymbol{v},t) = \sum_{e,i} N_s(\boldsymbol{x},\boldsymbol{v},t)\,, \quad \text{with } N_s(\boldsymbol{x},\boldsymbol{v},t) = \sum_{i=1}^{N_0} \delta[\boldsymbol{x}-\boldsymbol{X}_i(t)]\,\delta[\boldsymbol{v}-\boldsymbol{V}_i(t)]\,.$$

Each particle follows a trajectory given by Newton's law (kinetic theory)

$$\dot{\boldsymbol{X}}_i(t) = \boldsymbol{V}_i(t)\,, \quad m_s\dot{\boldsymbol{V}}_i(t) = q_s\boldsymbol{E}^m[\boldsymbol{X}_i(t),t] + q_s\boldsymbol{V}_i(t)\times\boldsymbol{B}^m[\boldsymbol{X}_i(t),t]\,,$$

The elective and magnetic fields follow Maxwell equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}^m = \rho^m\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B}^m = 0\,,$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}^m = -\partial_t\boldsymbol{B}^m\,, \quad \boldsymbol{\nabla}\times\boldsymbol{B}^m = \boldsymbol{J}^m + \partial_t\boldsymbol{E}^m\,,$$

with an average charge and current density given by the integrals over velocity (equivalent to a sum for the discrete description)

$$\rho^m = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v} N_s(\boldsymbol{x},\boldsymbol{v},t)\,, \quad \boldsymbol{J}^m = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\,N_s(\boldsymbol{x},\boldsymbol{v},t)\,,$$

The solution is a "spiky" number density N_s that follows Klimontovich equation

$$\partial_t N_s(\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\cdot\boldsymbol{\nabla}N_s + q_s\big(\boldsymbol{E}^m + \boldsymbol{v}\times\boldsymbol{B}^m\big)\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}N_s = 0\,,$$

Kinetic theory of plasmas

- Klimontovich equation is difficult to track for many particles and becomes computationally unaffordable to study plasmas.

- Average of Klimontovich equation over small parcels of fluid (which are small enough to describe local properties and large enough to contain a large number of particles) gives rise to the **plasma kinetic equation**

$$\partial_t f_s + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_s = -\frac{q_s}{m_s}\langle(\delta\boldsymbol{E}+\boldsymbol{v}\times\delta\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\,\delta N_s\rangle = (\,\mathrm{d}f_s/\,\mathrm{d}t)_c$$

where $f_s = \langle N_s \rangle$, such that $N_s = f_s + \delta N_s$.

Similarly, the electric and magnetic fields are split into an ensemble average and perturbations that characterize the collisions, $E^m = E + \delta E, \quad B^m = B + \delta B$

$$\partial_t \boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}\,, \qquad \partial_t \boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{J}\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_e\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,,$$

$$\rho(\boldsymbol{x},t) = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\, f_s(\boldsymbol{x},\boldsymbol{v},t)\,,$$

$$\boldsymbol{J}(\boldsymbol{x},t) = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\, f_s(\boldsymbol{x},\boldsymbol{v},t)\,.$$

Boltzmann/Vlasov equation

- Boltzmann equation describes the evolution of the distribution function f_s for each species s and contains a collision term $\Gamma(f_s)$ that characterizes the modifications of the distribution function due to collisions,

$$D_t f_s = \partial_t f_s + \left(\boldsymbol{v} \cdot \boldsymbol{\nabla}\right) f_s + \left(\frac{\mathbf{F_s}}{m_s} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}}\right) f_s = \Gamma(f_s)$$

- Boltzmann equation usually refers to neutral fluids (no electromagnetic forces) and Vlasov equation usually refers to collisionless plasmas (where collective effects are more relevant than particular collisions, e.g., infinite conductivity leading to ideal MHD).

- For binary collisions, the collision term can be described as an integral over scattering angles Ω with cross section $\sigma(\Omega)$ and the distribution functions of the two particles before (f_1', f_2') and after (f_1, f_2) the collision

$$\Gamma(f) = \int \mathrm{d}^3\boldsymbol{v}_2 \int \mathrm{d}\Omega\, \sigma(\Omega) |\boldsymbol{v}_1 - \boldsymbol{v}_2| (f_2' f_1' - f_2 f_1)\,,$$

Relativistic Boltzmann equation (Lichnerowicz and Marrot 1940)

Recommended literature: C. Cercignani and G. Medeiros Kremer, “The relativistic Boltzmann equation: Theory and applications”, 2002 and L. Rezzolla and O. Zanotti, “Relativistic hydrodynamics”, 2013

- Boltzmann equation can be described using a covariant formulation in the following way when extended to relativistic velocities

$$p^\mu D_\mu f_s + D_{p^\mu}(p^0 F^\mu f_s) = \Pi(f_s)$$

- This Boltzmann equation is also valid for massless species (like photons and the relativistic particles that dominate the primordial plasma content in the early Universe).

- The relativistic collision term is

$$\Pi(f_s) = \int \frac{\mathrm{d}^3\boldsymbol{p}_2}{p_2^0} \int \mathrm{d}\Omega\, \sigma(\Omega) \mathcal{F}(f_2' f_1' - f_2 f_1)$$

$$\mathcal{F} = p_1^0 p_2^0 g_\phi = p_1^0 p_2^0 \sqrt{(v_1 - v_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}$$

$\mathscr{F}$ is the collision flux and g_ϕ is the Møller’s relative speed

- This equation is equivalent to the classical Boltzmann equation,

$$\partial_t f_s + v^i \partial_i f_s + \partial_{p^i}(f_s F^i) = \int \mathrm{d}^3\boldsymbol{p}_2 \int \mathrm{d}\Omega\, \sigma(\Omega)\, g_\phi\, (f_2' f_1' - f_2 f_1)$$

Relativistic Boltzmann equation

- For external forces that do not depend on the momentum or that only depend on it through the Lorentz force, the Boltzmann equation becomes (exercise!)

$$\partial_t f_s + v^i \partial_i f_s + F^i \partial_{p^i} f_s = \int \mathrm{d}^3\boldsymbol{p}_2 \int \mathrm{d}\Omega\, \sigma(\Omega)\, g_\phi \left(f_2' f_1' - f_2 f_1\right)$$

- This Boltzmann equation is valid for non-degenerate species, its extension to degenerate species includes modifications to the collision terms for particles that obey quantum statistics (Uehling-Uhlenbeck equation)

$$f_2' f_1' - f_2 f_1 \to f_2' f_1' \left(1 + \varepsilon \frac{f_1 h^3}{g_s}\right)\left(1 + \varepsilon \frac{f_2 h^3}{g_s}\right) - f_2 f_1 \left(1 + \varepsilon \frac{f_1' h^3}{g_s}\right)\left(1 + \varepsilon \frac{f_2' h^3}{g_s}\right)$$

where h is Planck constant and g_s the degeneracy factor (2s + 1 for massive particles and 2s for massless particles, being s the spin number).

$$\varepsilon = \begin{cases} +1 & \text{for Bose–Einstein statistics,} \\ -1 & \text{for Fermi–Dirac statistics, and} \\ 0 & \text{for Maxwell–Boltzmann statistics.} \end{cases}$$

Average quantities (fluid variables)

- The distribution function allows us to compute the average values of any arbitrary quantity a at any location x and time t (ensemble average of particles with different velocities that compose the fluid parcel at a local position x)

$$\langle a\rangle_s(t,\boldsymbol{x}) = \frac{1}{n_s}\int a f_s(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}$$

Number density n_s

$$n_s(t,\boldsymbol{x}) = \int f_s(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}$$

- For the relativistic extension, $\mathrm{d}^3 v \to \mathrm{d}^3 p/p^0$

- The averaged quantities will allow us to describe fluid variables when we consider the multi-fluid and MHD descriptions in the following, for example, the fluid velocity is

$$\boldsymbol{u}_s(t,\boldsymbol{x}) = \frac{1}{n_s(t,\boldsymbol{x})}\int \boldsymbol{v} f_s(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}$$

Collisional invariants ψ

- The time evolution of the average value of an arbitrary quantity ψ (known as the production term due to collisions $\mathscr{P}$) is

$$\mathcal{P} = D_t\langle\psi\rangle = \int \psi \frac{\mathrm{d}^3\boldsymbol{p}_1}{p^0} D_t f_s = \int \psi \frac{\mathrm{d}^3\boldsymbol{p}_1}{p^0} \int \frac{\mathrm{d}^3\boldsymbol{p}_2}{p^0} \int \mathrm{d}\Omega\, \sigma(\Omega)\, \mathcal{F}(f_2' f_1' - f_2 f_1)$$

- It can be shown (exercise!) that the production term becomes

$$\mathcal{P} = \frac{1}{4} \int (\psi_1 + \psi_2 - \psi_1' - \psi_2') \frac{\mathrm{d}^3\boldsymbol{p}_1}{p^0} \int \frac{\mathrm{d}^3\boldsymbol{p}_2}{p^0} \int \mathrm{d}\Omega\, \sigma(\Omega)\, \mathcal{F}(f_2' f_1' - f_2 f_1)$$

- Therefore, it vanishes ($\mathscr{P} = 0$) for a collisional invariant ψ that satisfies

$$\psi_1 + \psi_2 = \psi_1' + \psi_2'$$

Collisional invariant and conservation equations

- It can further be shown (more difficult exercise!) that ψ is a collisional invariant iff

Theorem: *A continuous and differentiable function of class C^2 $\psi(p^\alpha)$ is a summational invariant if and only if it is given by:*

$$\psi(p^\alpha) = A + B_\alpha p^\alpha, \tag{2.56}$$

where A is an arbitrary scalar and B_α an arbitrary four-vector that do not depend on p^α.

- Examples of collisional invariants are the mass m, the momentum flux mv and the internal energy $\frac{1}{2}mw^2$, where $w = v - \langle v \rangle$ is the peculiar velocity. As an exercise, show that these quantities satisfy the property of collisional invariants.

- These collisional invariants can be used to derive the subrelativistic mass, momentum and energy conservation equations.

Conservation equations (non-relativistic Newtonian limit)

- Let us now multiply Boltzmann equation by a collisional invariant and average over the velocity space. After some algebra (exercise!) one finds a generic conservation equation

$$\boxed{\partial_t(n_s\langle\psi\rangle) + \boldsymbol{\nabla}\cdot(n_s\langle\psi\boldsymbol{v}\rangle) - n_s\langle\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\rangle - \frac{n_s}{m_s}\langle\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\psi\rangle = 0\ .}$$

- **Continuity equation** (number density/mass conservation) is found when $\psi = 1$ (or $\psi = m$ for massive particles), corresponding to the zeroth-moment of Boltzmann equation,

$$\partial_t n_s + \boldsymbol{\nabla}\cdot(n_s\boldsymbol{u}_s) = 0$$

$$\partial_t\rho_m + \boldsymbol{\nabla}\cdot\left(\rho_m\boldsymbol{u}\right) = 0$$

$$\boldsymbol{u}_s(t,\boldsymbol{x}) = \frac{1}{n_s(t,\boldsymbol{x})}\int \boldsymbol{v} f_s(t,\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}$$

Conservation equations (non-relativistic Newtonian limit)

- **Conservation of momentum** is found using the momentum flux $\psi = v_i$, corresponding to the first-moment of Boltzmann equation,

$$\partial_t(n_s \boldsymbol{u}_s) + \boldsymbol{\nabla} \cdot (n_s \boldsymbol{u}_s \boldsymbol{u}_s) + \frac{1}{m_s} \boldsymbol{\nabla} \cdot \boldsymbol{P}_s - n_s \boldsymbol{F} = 0 \,.$$

where the pressure tensor for a species s involves second-order moments of the distribution function

$$\frac{P_{ij}}{\rho_m} = \langle w_i w_j \rangle = \langle v_i v_j \rangle - u_i u_j = \frac{T_{ij}}{\rho_m} - u_i u_j$$

where $\rho_m = m_s n_s$ is the mass density.

Conservation equations (non-relativistic Newtonian limit)

- The pressure tensor is not a collisional invariant, hence its conservation equation (second-moment of the Boltzmann equation) contains a collision term in the right-hand-side (production term).

- However, the internal energy density, which corresponds to the trace of the pressure tensor,

$$\varepsilon_s = \frac{P_{ii}}{2m_s n_s} = \tfrac{1}{2}\langle \boldsymbol{w}^2 \rangle$$

is a collisional invariant and leads to the **conservation of energy**

$$\partial_t \big(n_s \varepsilon\big) + \boldsymbol{\nabla} \cdot \big(n_s \varepsilon_s \boldsymbol{u}_s\big) + \frac{\boldsymbol{\nabla} \cdot \boldsymbol{q}}{m_s} + \frac{P_{ij} S^{ij}}{m_s} = 0 \qquad S^{ij} = \tfrac{1}{2}\big(\partial^i u^j + \partial^j u^i\big)$$

which involves a third moment operator, the **heat flux tensor** Q_{ijl}, defined such that

$$q_i = Q_{ijj} = \langle w_i \, \boldsymbol{w}^2 \rangle$$

Conservation equations: The hierarchy problem

- We can get a system of infinite equations by taking each of the moments of the velocity multiplied by Boltzmann equation
- As we have observed, the dynamical equation for each moment involves a divergence of a higher-order moment. This is known as the hierarchy problem.
- To close the system, we need to consider an equation of state that relates, for example, the pressure tensor and the heat flux to the fluid variables (number/mass density, fluid velocity and internal energy density).
- To summarize, the conservation equations for a plasma under the Lorentz force are

$$\partial_t n_s + \boldsymbol{\nabla}\cdot(n_s \boldsymbol{u}_s) = 0\,,$$
$$\partial_t(n_s \boldsymbol{u}_s) + \boldsymbol{\nabla}\cdot(n_s \boldsymbol{u}_s \boldsymbol{u}_s) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}_s + \frac{q_s n_s}{m_s}\big(\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B}\big)\,,$$
$$\partial_t(n_s \varepsilon_s) + \boldsymbol{\nabla}\cdot(n_s \varepsilon_s \boldsymbol{u}_s) = -\boldsymbol{\nabla}\cdot\boldsymbol{q}_s - P_{s,ij} S_s^{ij}\,.$$

Then, these equations can be solved together with Maxwell equations,

$$\partial_t \boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}\,, \qquad \partial_t \boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{J}\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_e\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,,$$

where the charge and current densities are

$$\rho_e = \sum_s q_s n_s\,, \qquad \boldsymbol{J} = \sum_s q_s n_s \boldsymbol{u}_s\,.$$

Conservation equations (relativistic)

- We can also find the relativistic version of the conservation equations multiplying the relativistic Boltzmann equation with the four-momentum p^μ and taking the average.

- The conservation equations up to first moment are

$$N_s^\mu = \int p^\mu f_s \frac{\mathrm{d}^3\boldsymbol{p}}{p^0}$$

$$D_\mu N_s^\mu = 0\,, \quad D_\mu T_s^{\mu\nu} = m_s \int F^\nu f_s \frac{\mathrm{d}^3 p}{p^0},$$

$$T_s^{\mu\nu} = \int p^\mu p^\nu f_s \frac{\mathrm{d}^3\boldsymbol{p}}{p^0}$$

obtained using $\psi = 1$ and p^μ, respectively.

- This system of equations correspond to the relativistic version of the mass (density), momentum flux and energy conservation.

- The integral over the Lorentz force reduces to

$$D_\mu T^{\mu 0} = 0\,, \qquad D_\mu T_s^{\mu i} = q_s n_s (E + u_s \times B)^i$$

Fluids near local thermal equilibrium (LTE)

- If we consider small Knudsen numbers, then we can in general assume that each fluid parcel contains a large number of collisions that drives it to local thermal equilibrium, which allows us to consider a perturbation of the distribution function around equilibrium,

$$f(t, \boldsymbol{x}, \boldsymbol{v}) = f_0(t, \boldsymbol{x}, \boldsymbol{v}) + \mathrm{Kn} f_1(t, \boldsymbol{x}, \boldsymbol{v}) + \mathcal{O}(\mathrm{Kn}^2) \qquad \mathrm{Kn} = l_{\mathrm{mfp}}/l \ll 1$$

- A perfect fluid is found when the distribution function is exactly given by the equilibrium one, i.e., when $f_1 = 0$ (zero-order hydrodynamics).

- LTE distributions are stationary solutions of the Boltzmann equation, such that the collision terms vanishes. The solution to this equation for non-relativistic Newtonian fluids is the Maxwell-Boltzmann distribution (exercise)

$$f_1' f_2' - f_1 f_2 = 0 \quad \Longrightarrow \quad \boxed{f_0(\boldsymbol{v}) = n\left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{\frac{3}{2}} \exp\left(-\frac{m|\boldsymbol{v}-\boldsymbol{u}|^2}{2k_{\mathrm{B}}T}\right) .}$$

Fluids near local thermal equilibrium (LTE)

- Perfect fluids: for perfect fluids (ideal gas), the pressure tensor becomes isotropic and the heat flux vanishes.

$$P_{ij} = p\delta_{ij} = \frac{2}{3}\rho_m \varepsilon \delta_{ij}$$

$$\boldsymbol{q} = \tfrac{1}{2}\rho_m \langle w^2 \boldsymbol{w} \rangle = 0\,.$$

$$\partial_t \boldsymbol{u}_s + \big(\boldsymbol{u}_s \cdot \boldsymbol{\nabla}\big)\boldsymbol{u}_s + \frac{1}{\rho_m}\boldsymbol{\nabla} p = \frac{\boldsymbol{F}}{m_s}$$

Euler equation

$$\partial_t \varepsilon_s + \big(\boldsymbol{u}_s \cdot \boldsymbol{\nabla}\big)\varepsilon_s + \frac{p}{\rho_m}\boldsymbol{\nabla} \cdot \boldsymbol{u}_s = 0$$

Fluids near local thermal equilibrium (LTE)

- Imperfect fluids (first-order hydrodynamics): taking the first term in the perturbative expansion around Knudsen number (Chapman-Enskog theory), the pressure tensor includes an anisotropic term, which can be described by the Navier-Stokes viscosity, and the heat flux is described by Fourier's law (see Roper Pol & Midiri 2025 for details)

$\Pi^{ij} = 2\,\nu\,\sigma^{ij} + \zeta\,\theta\,\delta^{ij}$ is the deviatoric stress tensor, $P_{ij} = p\delta_{ij} - \Pi_{ij}$

$$\boldsymbol{q} = -\kappa \boldsymbol{\nabla} T$$

$$\sigma^{ij} = S^{ij} - \tfrac{1}{3}\theta\,\delta^{ij}$$

$$\partial_t \boldsymbol{u}_s + (\boldsymbol{u}_s \cdot \boldsymbol{\nabla})\boldsymbol{u}_s = -\frac{1}{\rho_m}\boldsymbol{\nabla} p + \frac{\nu}{\rho_m}\boldsymbol{\nabla}^2 \boldsymbol{u}_s + \frac{1}{\rho_m}\left(\zeta + \tfrac{1}{3}\nu\right)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u}_s + \frac{q_s n_s}{\rho_m}(\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B})$$

Navier-Stokes equation

$$\partial_t \varepsilon + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\varepsilon = \frac{1}{\rho_m}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T) - \frac{p}{\rho_m}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{\lambda}{\rho_m}(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2 + \frac{2\nu}{\rho_m}S_{ij}S_{ij}$$

$$\lambda = \zeta - \tfrac{2}{3}\nu$$

Fluids near local thermal equilibrium (LTE)

- For relativistic fluids, the solutions to the stationary Boltzmann equation lead to the Maxwell-Jüttner (1911) distribution and to the Bose-Einstein and Fermi-Dirac distribution functions (see literature for detailed derivations)

Maxwell-Jüttner

$$f^{(0)} = \frac{g_\mathrm{s}}{h^3} e^{\frac{\mu E}{kT} - \frac{U^\alpha p_\alpha}{kT}}$$

Fermi-Dirac (+) and Bose-Einstein (-)

$$f^{(0)} = \frac{g_\mathrm{s}/h^3}{e^{-\frac{\mu E}{kT} + \frac{U^\alpha p_\alpha}{kT}} \pm 1}$$

- Relativistic perfect fluids: the stress-energy tensor of perfect fluids can be described as

$$T^{\mu\nu} = (p + \rho) U^\mu U^\nu + p \eta^{\mu\nu}, \qquad \text{where } U^\mu = N^\mu / n$$

(-+++) signature

- Relativistic imperfect fluids: first-order hydrodynamics, also known as classical irreversible thermodynamics (CIT), suffer from causality problems in the relativistic limit and are not a satisfactory description of dissipative effects, so it is in general required to consider second-order theories (extended irreversible thermodynamics, EIT); see Roper Pol & Midiri (2025) review, Romatschke (2009) and Rezzolla's book.

Application to the early Universe

- In the early Universe, we will consider a primordial plasma composed of radiation massless particles.

- In the limit of massless particles, independently of their distribution function (Fermi-Dirac for fermions or Bose-Einstein for bosons), it can be shown (exercise! hint: see Rezzolla's book) that the isotropic pressure is one third of the total energy density (ultra-relativistic equation of state):

$$p = \frac{1}{3}\,\rho$$

- This equation of state allows us to close the system of fluid equations in equilibrium, which can be solely described by the conservation law

$$D_\mu T_s^{\mu\nu} = 0$$

Single fluid. MHD description

- Finally, we are ready to combine the multi-fluid equations into a single-fluid description (MHD).

- Let us start with plasmas composed of non-relativistic particles containing both electrons and ions. Until now, we have considered collisional invariants when dealing with collisions between the same species. However, for interacting particles, we will get an additional collision term in the momentum equation of each species

$$\boldsymbol{K}_s = m_s \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v}(\,\mathrm{d}f_s/\,\mathrm{d}_t)_c$$

- This term corresponds to the modifications of the average velocity (momentum) of particles of species s due to collisions with other particles and should satisfy the property

$$\sum_s K_s = K_e + K_i = 0$$

Single fluid. MHD description (non-relativistic)

- Therefore, we can now sum up the continuity and momentum equations to find the following conservation equations for the combined single conducting fluid (MHD description),

$$\boxed{\partial_t \rho_m + \boldsymbol{\nabla} \cdot (\rho_m \boldsymbol{u}) = 0\,.} \qquad \boxed{\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} P + \boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{F}/m + \Pi_{ij}\,,}$$

where we have defined the single-fluid mass and charge density, current density, total pressure, and velocity

$$\rho_m = m_e n_e + m_i n_i \approx m_i n_i\,, \quad \rho_e = q_e n_e + q_i n_i \approx e(n_i - n_e)\,,$$

$$\boldsymbol{u} = \frac{1}{\rho_m}(m_i n_i \boldsymbol{u}_i + m_e n_e \boldsymbol{u}_e)\,,$$

$$\boldsymbol{J} = q_i n_i \boldsymbol{u}_i + q_e n_e \boldsymbol{u}_e\,, \quad P = P_i + P_e\,.$$

Single fluid. MHD description

- In the relativistic limit, we find

$$D_\mu T^{\mu\nu} = f^\nu_{\rm Lor}, \qquad\qquad T^{\mu\nu} = \sum_s T^{\mu\nu}_s$$

where $f^\nu_{\rm Lor} = (E \cdot J, \rho_e E + J \times B)$ is the Lorentz four-force and is is equivalent to the divergence of the electromagnetic stress-energy tensor, since this conservation law has to be compatible with Bianchi identities according to GR (see Roper Pol & Midiri 2025).

- On the other hand, if one combines the momentum equations of different species (see Nicholson for details), the generalized Ohm's law can be found, which after a few simplifications it leads to

$$J^\mu = \rho_e U^\mu + \sigma E^\mu \qquad\qquad \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

Covariant formulation

Non-relativistic Ohm's law

Single fluid. MHD description in an expanding Universe

- If two metric tensors can be transformed into each other via a Weyl transformation

$$g^{\mu\nu} = \Omega^2(x^\mu)\,\tilde{g}^{\mu\nu}.$$

then the equations of motion in the metric g can be transformed to the equations of motion in the metric $\tilde{g}$ in the following way

$$D_\mu T^{\mu\nu} = 0 \Leftrightarrow D_{\tilde{\mu}} \tilde{T}^{\mu\nu} + \Omega^{-2}\,\tilde{T}^\mu{}_\mu\,\partial^\nu \ln\Omega = 0 \qquad \tilde{T}^{\mu\nu} = \Omega^{-6}\,T^{\mu\nu}$$

This shows that the equations in an expanding Friedmann-Robertson-Walker metric tensor can be expressed as those in flat space-time (Minkowski) when considering conformal time, such that

$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu}$$

The final set of equations after conformal transformation become

$$\partial_\mu \tilde{T}^{\mu 0} + \tilde{T}\,\mathcal{H} = 0\,, \qquad \partial_\mu \tilde{T}^{\mu i} = 0\,. \qquad \tilde{T} = a^4\,T = 3\tilde{p} - \tilde{\rho}$$

Single fluid. MHD description in an expanding Universe

Relativistic MHD equations in an expanding Universe (Roper Pol & Midiri 2025)

$$\partial_\tau \ln \tilde\rho + \frac{1+c_{\rm s}^2}{1-c_{\rm s}^2 u^2}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_{\rm s}^2}{1-c_{\rm s}^2 u^2}\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde\rho = \frac{1+u^2}{1-c_{\rm s}^2 u^2}\,(1-3c_{\rm s}^2)\,\mathcal{H} + \frac{1}{1-c_{\rm s}^2 u^2}\left[\frac{\tilde f^0_{\rm tot}}{\tilde\rho}(1+u^2) - 2\,\frac{\boldsymbol{u}\cdot\tilde{\boldsymbol{f}}_{\rm tot}}{\tilde\rho}\right], \qquad (1.2a)$$

Conservation of $T^{\mu\nu}$

$$D_\tau\boldsymbol{u} = \frac{\boldsymbol{u}}{(1-c_{\rm s}^2u^2)\gamma^2}\left[c_{\rm s}^2\,\boldsymbol{\nabla}\cdot\boldsymbol{u} + c_{\rm s}^2\frac{1-c_{\rm s}^2}{1+c_{\rm s}^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\tilde\rho + (3c_{\rm s}^2-1)\,\mathcal{H} - \frac{\tilde f^0_{\rm tot}}{\tilde\rho} + \frac{2c_{\rm s}^2}{1+c_{\rm s}^2}\frac{\boldsymbol{u}\cdot\tilde{\boldsymbol{f}}_H}{\tilde\rho}\right] - \frac{c_{\rm s}^2}{1+c_{\rm s}^2}\frac{\boldsymbol{\nabla}\ln\tilde\rho}{\gamma^2} + \frac{1}{1+c_{\rm s}^2}\,\frac{\tilde{\boldsymbol{f}}_{\rm tot}}{\tilde\rho\gamma^2}\,, \qquad (1.2b)$$

$$\partial_\tau\tilde{\boldsymbol{E}} = \boldsymbol{\nabla}\times\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{J}}\,, \qquad \partial_\tau\tilde{\boldsymbol{B}} = -\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}\,, \qquad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}} = \tilde J^0\,, \qquad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0\,,$$

Maxwell equations

$$\tilde J^i = \tilde\rho_e\boldsymbol{u} + \tilde\sigma(\tilde{\boldsymbol{E}} + \boldsymbol{u}\times\tilde{\boldsymbol{B}})$$

Ohm's law