Relativistic magnetohydrodynamics in the early Universe. $\mathcal{C}$osmo-$\mathcal{L}$attice school

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This lecture on “Relativistic magnetohydrodynamics in the early Universe” is part of the $\mathcal{C}$osmo-$\mathcal{L}$attice school and uses material from the “Numerical simulations of magnetohydrodynamics in the early Universe” course given at EPFL by A. Roper Pol and J. Schober on Spring 2024; see the course website for the extended course material. The lectures are password protected. To access them, please contact `alberto.roperpol@unige.ch`

The theoretical description of the relativistic MHD equations in an expanding Universe can be found in A. Roper Pol and A. S. Midiri, “Relativistic magnetohydrodynamics in the early Universe,” arXiv:2501.05732. This reference contains a review on this topic as well as new developments that are currently being implemented and publicly available on Pencil Code and under development on $\mathcal{C}$osmo-$\mathcal{L}$attice with D. G. Figueroa, K. Marschall, and A. S. Midiri.

Contents

1 Introduction

Magnetohydrodynamics (MHD) is the study of the dynamics of a plasma and the electromagnetic fields. MHD is used to study fluids that are partially or fully ionized and can carry density currents. These density currents yield magnetic fluctuations that at the same time drive the fluid particles via the Lorentz force. In such situations, the fluid motion induced by magnetic fields needs to be included in the fluid's equation of momentum conservation. At the same time, Maxwell equations determine the dynamics of the electromagnetic fields in the presence of currents. The resulting dynamics is then governed by the fluid conservation laws: conservation of mass, momentum, and energy, together with Maxwell equations.

For neutral uncharged fluids, the fluid description allows to represent scales larger than the mean free path under the assumption that quantum effects are negligible, i.e., scales larger than the de Broglie wavelength. A system of N particles can be treated like classical particles when their typical separation l is much larger than the de Broglie wave length of each particle, such that $R = \lambda_{\rm DB}/l \ll 1$. In this case, the wavefunctions of each particle are separated and quantum interference is not relevant, such that the position and velocity of the particles can be described by classical mechanics. This is known as the Ehrenfest's theorem. When the system contains a large number of particles N over a length scale L much larger than l, characterized by low Knudsen number $\mathrm{Kn} = l/L \ll 1$, then the collective dynamics of the system can be approximated by a continuum (or fluid) description. Each patch of the fluid contains a large amount of particles but are small enough to guarantee homogeneity within the patch, which is given the same velocity on average and is assumed to be in thermodynamic equilibrium.

For a plasma, composed of freely charged particles, using MHD we require the fluid scales to be larger than the plasma scale lengths as the ion gyroradius. At smaller scales, kinetic theory becomes necessary to describe the plasma.

Plasmas can be found at laboratories, confined by strong magnetic fields, and this corresponds to an extremely active topic of research, in particular related to nuclear fusion. However, plasmas are more abundant outside our planet, in stars, galaxies, and even in the early Universe.

Validity of MHD

In the first place, we can consider a distribution of charged particles formed by a number of different species s, for example, ions and electrons. The most elemental theory once quantum and relativistic effects can be ignored is kinetic theory, which describes the motion of charged particles in the presence of electric and magnetic fields.

In the case of charged particles, electrostatic forces will generate a Debye screening around a charged particle, which produces a electrostatic potential,

$$\phi = \frac{q}{r} e^{-r/\lambda_{\rm D}} \,, \tag{1}$$

where the Debye length is

$$\lambda_{\rm D}^{-2} = \lambda_e^{-2} + \lambda_i^{-2} \,, \quad \lambda_{e,i} = \sqrt{\frac{T_{e,i}}{n_0\, e^2}} \,, \tag{2}$$

for a density distribution of particles n_0. This assumes that electrons and ions have reached thermal equilibrium separately. Therefore, the potential of an individual particle decays much rapidly that $1/r$ due to the effective cloud of particles with opposite charge that screen the test charge particle. This shows the collective behavior of a plasma when we consider distances $l \gg \lambda_{\rm D}$. This can be quantified by the plasma parameter of species s, obtained by comparing the average potential energy Φ of a particle due to its neares neighbor,

$$\Phi \sim \frac{e^2}{r} \sim n_0^{1/3} e^2 \,, \tag{3}$$

with the particle's kinetic energy $\frac{1}{2} m_s \langle v^2 \rangle = \frac{3}{2} T_s$. Comparing this and requesting that kinetic energy is larger than electrical potential energy, we find

$$\Lambda_s \equiv n_0 \lambda_s^3 \gg 1 \,. \tag{4}$$

Hence, the requirement for a plasma approximation is that the plasma parameter, i.e., the number of particles of species s in a box of Debye length size is large.

The Debye length is also related to the plasma frequency. Let us consider two slabs of electrons and ions, with a difference in size δ, then the electric charge of the ion's slab into the electron's slab leads to a harmonic oscillator for δ with a frequency,

$$\omega_e^2 = \frac{n_0 e^2}{m_e}\,, \tag{5}$$

usually much larger than that for ions, as $m_i \gg m_e$. The Debye length is $\lambda_s = v_s/\omega_s$, being $v_s =$ the thermal speed.

Another important length scale in the plasma is the gyroradius, or Larmor radius, corresponding to the oscillatroy motion induced by a magnetic field in its perpendicular direction. The mean gyroradius is

$$r_s = \frac{v_s m_s}{q_s B_0}\,. \tag{6}$$

In general, we will consider MHD to be a valid theory when all fluid scales are larger than the Debye length and the Larmor radius (such that we can ignore local plasma effects) and larger than the de Broglie wavelength (allowing us to ignore quantum effects). The plasma scales usually play the role of the mean free path considered for the validity of the fluid approach when dealing with plasmas. For radiation particles, the gyroradius vanishes and hence, this will not incorporate a relevant scale for determining the validity of the MHD description.

About collisions

We have seen that the Debye screening avoids a charged particle to effectively "collide" with particles further than their Debye length, so we can approximately consider that each particle undergoes Λ simultaneous Coulomb collisions. Even though many collisions can occur (as Λ is large) in a plasma, each collision is weak, so the total collective effect of collisions might be weak. Note that the requirement of the potential energy to be much smaller than the thermal energy implies $\Lambda \gg 1$.

The importance of collisions can be estimated using the collision frequency, the inverse of the time it takes for a particle to suffer a collision. This can be estimated to be

$$\frac{\nu_c}{\omega_e} \approx \frac{\ln \Lambda}{2\pi \Lambda_e}\,. \tag{7}$$

This estimates the number of collisions between electrons and ions and will give rise to the conductivity. The addition of collisions among ions will correct this value. Furthermore, the mean-free path of the ions l_i will determine the viscosity $\nu \sim v_t l_i$ with v_t being determined by their random velocity. Hence, the viscosity will be a single fluid effect that will become relevant after we derive the single-fluid equation (MHD) and it can be understood in an analogous way to the viscosity obtained in hydrodynamics for a fluid composed of neutral particles (see Brandenburg and Subramanian 2005, and Spitzer's book for estimates of these coefficients, and Arnold, Moore, Yaffe, 2000. for estimates of these coefficients in the early Universe). Thus, for large Λ, the effect of collisions is much smaller than that of collective effects, for example, the electron frequency. For the description of collisions and estimation of transport coefficients in the early Universe.

2 Kinetic theory of plasmas

2.1 Klimontovich equation

Let us now consider the number density of charged particles

$$N(\boldsymbol{x},\boldsymbol{v},t) = \sum_{e,i} N_s(\boldsymbol{x},\boldsymbol{v},t)\,, \quad \text{with } N_s(\boldsymbol{x},\boldsymbol{v},t) = \sum_{i=1}^{N_0} \delta[\boldsymbol{x} - \boldsymbol{X}_i(t)]\,\delta[\boldsymbol{v} - \boldsymbol{V}_i(t)]\,. \tag{8}$$

For any plasma composed of different particles, we can extend this description to each species (for example to quarks, gluons, ..., in the early Universe). If we know the position and velocity of all particles i, then we can solve the equations of motion for each particle

$$\dot{\boldsymbol{X}}_i(t) = \boldsymbol{V}_i(t)\,, \quad m_s\dot{\boldsymbol{V}}_i(t) = q_s\boldsymbol{E}^m[\boldsymbol{X}_i(t),t] + q_s\boldsymbol{V}_i(t)\times\boldsymbol{B}^m[\boldsymbol{X}_i(t),t]\,, \tag{9}$$

where the electric and magnetic fields are produced by the charged particles following Maxwell equations,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}^m = \rho^m\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B}^m = 0\,, \tag{10}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}^m = -\partial_t\boldsymbol{B}^m\,, \quad \boldsymbol{\nabla}\times\boldsymbol{B}^m = \boldsymbol{J}^m + \partial_t\boldsymbol{E}^m\,, \tag{11}$$

and

$$\rho^m = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v} N_s(\boldsymbol{x},\boldsymbol{v},t)\,, \quad \boldsymbol{J}^m = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\,N_s(\boldsymbol{x},\boldsymbol{v},t)\,, \tag{12}$$

are the charge and current densities. Under this description, charged particles are treated as discrete point particles. Then, by conservation of the number density of particles, one can find the exact Klimontovich equation of kinetic plasmas,

$$\partial_t N_s(\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\cdot\boldsymbol{\nabla}N_s + q_s\big(\boldsymbol{E}^m + \boldsymbol{v}\times\boldsymbol{B}^m\big)\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}N_s = 0\,, \tag{13}$$

which naively might look similar to the collisionless Boltzmann equation for fluids, but now this is for a discrete "spikey" number density. This is an exact equation that allows us to solve for the position of a system of particles, together with Maxwell equations. However, this is extremely costly as it requires to follow the orbit of each of the particles in the system.

2.2 Plasma kinetic equation

We now consider the probabilty of finding a particle in a small volume $\Delta\boldsymbol{x}\Delta\boldsymbol{v}$ of phase space centered at $(\boldsymbol{x}, \boldsymbol{v})$ and average $\langle N_s(\boldsymbol{x}, \boldsymbol{v}, t)\rangle$ around this point, leading to a smooth function $f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ that contains multiple particles. This corresponds to the distribution function of the particle species s. We can now consider expansions of the number density $N_s = f_s + \delta N_s$ and the resulting electric and magnetic fields $\boldsymbol{E}^m = \boldsymbol{E} + \delta\boldsymbol{E}$ and $\boldsymbol{B}^m = \boldsymbol{B} + \delta\boldsymbol{B}$ in the Klimontovich equation to find

$$\partial_t f_s + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \frac{q_s}{m_s}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_s = -\frac{q_s}{m_s}\langle(\delta\boldsymbol{E} + \boldsymbol{v}\times\delta\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\,\delta N_s\rangle = (\,\mathrm{d}f_s/\,\mathrm{d}t)_c\,, \tag{14}$$

where the right hand side corresponds to terms that are sensitive to the discrete nature of the particles, i.e., collisions, while the left hand side to collective effects. This is the **plasma kinetic equation**. When we can ignore the effects of collisions we find Vlasov equation (analogous to collisionless Boltzmann equation), which is usually solved to study plasma instabilities (e.g., Weibel instabilities).

Then, the charge and current densities are obtained as

$$\rho(\boldsymbol{x}, t) = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\, f_s(\boldsymbol{x}, \boldsymbol{v}, t)\,, \tag{15}$$

$$\boldsymbol{J}(\boldsymbol{x}, t) = \sum_{e,i} q_s \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\, f_s(\boldsymbol{x}, \boldsymbol{v}, t)\,. \tag{16}$$

BBGKY hierarchy

Before moving to fluid equations, let us consider the Liouville equation, which is equivalent to Klimontovich equation when we deal with a system of N particles and we consider f as the joint distribution function of all the particles in the system $f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, t)$. Then, we can show that the equation for the marginal distribution function obtained by the integration over $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$ space for all $i > k$ is

$$\partial_t f_k + \sum_{i=1}^{k} \boldsymbol{v}_i\cdot\boldsymbol{\nabla}_{\boldsymbol{x}_i} f_k + \sum_{i=1}^{k}\sum_{j=1}^{k} \boldsymbol{a}_{ij}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}_i} f_k + \frac{N_0 - k}{V}\sum_{i=1}^{k}\int \mathrm{d}\boldsymbol{x}_{k+1}\,\mathrm{d}\boldsymbol{v}_{k+1}\boldsymbol{a}_{i,k+1}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}_i} f_{k+1} = 0\,. \tag{17}$$

This is the BBGKY (Bogoliubov, Born and Green, Kirkwood, Yvon) hierarchy equation that couples each f_k to f_{k+1}. This equation is as complicated as the original Liouville equation, but we can consider only the distribution function of one particle after averaging over all the other particles of the system, and that of two particles (the latter corresponds to the effect of interactions between two particles, or binary collisions),

$$\partial_t f(\boldsymbol{x}, \boldsymbol{v}, t) + \boldsymbol{v}\cdot\boldsymbol{\nabla} f(\boldsymbol{x}, \boldsymbol{v}, t) + \frac{N_0 - 1}{V}\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\boldsymbol{a}_{12}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_2(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = 0\,. \tag{18}$$

We can express f_2 as

$$f_2(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = f(\boldsymbol{x}, \boldsymbol{v}, t)\, f(\boldsymbol{x}_2, \boldsymbol{v}_2, t) + g(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{x}_2, \boldsymbol{v}_2, t)\,, \tag{19}$$

which is the first step of the Mayer cluster expansion. The function g corresponds to the correlation function between particles 1 and 2. We can see that the integral over $\boldsymbol{x}_2$ and $\boldsymbol{v}_2$ of the acceleration $\boldsymbol{a}_{12}$ exerted by particle 2 over particle 1, corresponds to the acceleration $\boldsymbol{a}$,

$$\partial_t f + \boldsymbol{v}_1 \cdot \boldsymbol{\nabla} f + \boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f = -n_0 \int \mathrm{d}\boldsymbol{x}_2\, \mathrm{d}\boldsymbol{v}_2\, \boldsymbol{a}_{12} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} g\,. \tag{20}$$

As the function $g = f_2 - ff$, we can go to next order in the BBGKY expansion and derive an equation for f_2 where 3 particle interactions will appear (f_3), corresponding to next term in the Mayer cluster expansion, $f_3(123) = f(1)f(2)f(3) + f(1)g(23) + f(2)g(13) + f(3)g(12) + h(123)$. Ignoring $h(123)$ and inserting f_3 in the BBGKY expansion [see Eq. (17)] allows to find an additional equation for g. Note that h is of higher order in Λ. This treatment allows to solve for all possible binary collisions in the system but it is untractable for many practical applications. To simplify the system, we can assume that the function g relaxes on a time scale very short compared to the time scale on which f relaxes. This can be understood considering the incorporation of an electron in our plasma, then leading to a time for the other electrons to adjust the time it would take them to have a collision with the new electron, which is proportional to $\lambda_e/v_e \sim 1/\omega_e$. On the other hand, the time it would require f_1 to change because of collisions is $\sim \Lambda/\omega_e \gg 1/\omega_e$. This is known as ***Bogoliubov's hypothesis*** and a similar concept was already introduced in Chapman Enskog theory to approximate the heat conductivity and the viscosity coefficients. Note that Chapman Enskog theory also requires an expansion of the distribution function in Knudsen number, $\mathrm{Kn} \equiv l_{\mathrm{mfp}}/L$,

$$f = f^{(0)} + \mathrm{Kn}\, f^{(1)} + \mathrm{Kn}^2\, f^{(2)} + \dots\,, \tag{21}$$

where $f^{(0)}$ is the Maxwellian distribution and $f^{(1)}$ the first term in the expansion that includes the effects of binary collisions.

Under the 2-BBGKY hierarchy expansion and Bogoliubov's theory, one can find Lenard-Balescu equation that allows to compute f by including binary collision terms,

$$\frac{\partial f}{\partial t} = -\frac{n_0}{m_e^2} \nabla_{\boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{v}' \boldsymbol{k}\boldsymbol{k} \frac{\phi^2(k)}{|\epsilon(\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v})|^2} \delta[\boldsymbol{k}(\boldsymbol{v} - \boldsymbol{v}')][f(\boldsymbol{v})\nabla_{\boldsymbol{v}} f(\boldsymbol{v}') - f(\boldsymbol{v}')\nabla_{\boldsymbol{v}} f(\boldsymbol{v})]\,, \tag{22}$$

with $\phi(k) = e^2/k^2$ being the Fourier transform of the Coulomb potential and the dielectric function being

$$\epsilon(\boldsymbol{k}, \omega) = 1 + \frac{\omega_e^2}{k^2} \int \mathrm{d}\boldsymbol{v} \frac{\boldsymbol{k} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f(\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,. \tag{23}$$

Many plasma collisional phenomena can be studied using Lenard-Balescu equation and it is relevant to study many of the plasma instabilities in collisional plasmas. An interesting aspect of this equation is that its solution tends to a Maxwell distribution when $t \to \infty$ and conserves the particle density, the momentum and the kinetic energy. It can be shown that Lenard-Balescu equation can be expressed as a Fokker-Planck equation (known as its Landau form),

$$\partial_t f(\boldsymbol{v}, t) = -\boldsymbol{\nabla}_{\boldsymbol{v}} \cdot [\boldsymbol{A}(f(\boldsymbol{v})] + \tfrac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\nabla}_{\boldsymbol{v}} : [\boldsymbol{B} f(\boldsymbol{v})] \,, \tag{24}$$

being $\boldsymbol{A}$ and $\boldsymbol{B}$ the coefficients of dynamic friction and diffusion.

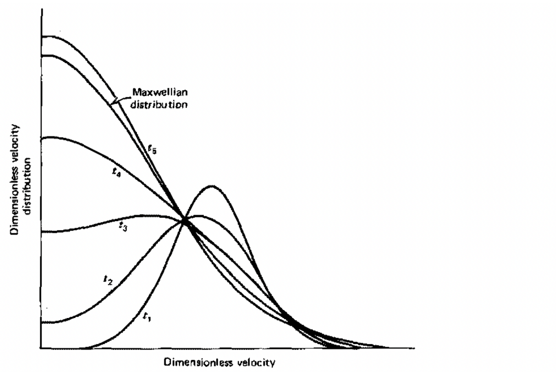

Fig. 5.2 Time evolution of a spherically symmetric electron distribution function as obtained from a numerical solution of the Landau form of the Fokker-Planck equation (5.32) by MacDonald et al. [7].

3 Boltzmann and Vlasov equations

The dynamics of the distribution function f are described by Boltzmann equation when the system is composed of electrically neutral particles. When the plasma is only subject to Coulomb forces, the dynamics is described by Vlasov-Maxwell equation. The total number of particles of the species s is given by integrating over the position and velocity 6-dimensional space,

$$N_s(t) = \int f(t, \boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{v} \tag{25}$$

In the absence of collisions, the Boltzmann equation for a system of particles with mass m and an external force $\boldsymbol{F}$ is

$$\frac{\mathrm{d}f_s(t, \boldsymbol{x}, \boldsymbol{v})}{\mathrm{d}t} = \partial_t f_s + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big) f_s + \left(\frac{\mathbf{F_s}}{m_s} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}}\right) f_s = 0 \,, \tag{26}$$

where $\boldsymbol{\nabla}_{\boldsymbol{v}} = \partial/\partial v^i$, and it represents the conservation of the distribution function f, i.e., the advection in position due to the motion of particles with velocity $\boldsymbol{v}$ and the advection in velocity due to the acceleration induced by the external force $\boldsymbol{F}_s = q_s(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. Collisions introduce a modification of the distribution function, leading the the Boltzmann equation

$$\partial_t f_s + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big) f_s + \left(\frac{\mathbf{F_s}}{m_s} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}}\right) f_s = \Gamma(f_s) \;, \tag{27}$$

where $\Gamma = \mathrm{d}f_s/\mathrm{d}t$ due to collisions.

The simplest case considers only binary collisions between particles at velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. In this case, the collision term is

$$\Gamma(f) = \int \mathrm{d}^3\boldsymbol{v}_2 \int \mathrm{d}\Omega\, \sigma(\Omega)|\boldsymbol{v}_1 - \boldsymbol{v}_2|(f_2' f_1' - f_2 f_1)\,, \tag{28}$$

where prime determines the distribution function after the collision and $f_{1,2} = f(t, \boldsymbol{x}, \boldsymbol{v}_{1,2})$. $\sigma(\Omega)$ is the differentiable cross section over the solid angle Ω of the short-range interaction responsible for the collisions. In a real plasma, one needs to consider the interactions between all the different particles, which can be extremely challenging. In general, we will only consider a perturbation expansion around the distribution function in local thermal equilibrium and use an effective description of the transport coefficients.

The average value of any quantity a is

$$\langle a\rangle(t, \boldsymbol{x}) = \frac{1}{n}\int a f(t, \boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v}, \tag{29}$$

where n is the number density,

$$n(t, \boldsymbol{x}) = \int f(t, \boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v}\ . \tag{30}$$

In general, collision effects are less important when dealing with charged plasmas than with fluids composed by neutral particles due to effects like Debye screening and the confinement of particles around magnetic fields. We have already argued that $\Lambda \gg 1$ is in general large for most of plasma applications. Hence, we will ignore their effects for now and only incorporate them after we have derived the fluid equations via the inclusion of viscous effects and electrical conductivity. This will allow us to consider phenomena with characteristic frequency $\omega \gg \omega_e/\Lambda$, i.e., short time-scales compared with collisions.

The resulting equation for plasmas when we neglect collisions is Vlasov equation,

$$\partial_t f_s + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \frac{q_s}{m_s}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_s = 0\,, \tag{31}$$

This equation is usually solved to study collisionless plasma instabilities (e.g., Weibel instability), linear waves (electrostatic waves like Langmuir waves, ion-acoustic waves, Bernstein modes), nonlinear waves (BGK modes, and damping effects (e.g., Landau damping, cyclotron damping). Open-source codes like Runko are available to solve for Vlasov equations using particle-in-cell simulations. Within Vlasov theory one can study many plasma instabilities, waves, Landau damping, etc. that cannot be incorporated in a fluid description, generally due to the $\boldsymbol{v}$ dependence of the distribution functions. Furthermore, many plasma waves that can be studied in two-fluid theories already appear in kinetic theory. These plasma effects will only be relevant for small frequencies but not close to zero, as this will correspond to stationary situations (infinitely small collision time) that can again be studied once we incorporate an MHD fluid description (similar as we did with hydrodynamics).

The covariant formulation of Boltzmann equation was first obtained by Lichnerowicz and Marrot (1940),

$$p^\mu D_\mu f_s + m D_{p^\mu}(F^\mu f_s) = \Pi(f_s)\,, \tag{32}$$

where p^μ is the four-momentum and Π the relativistic extension of the collision term, taking into account interactions of particles with different momentum p^μ. This equation allows us to extend the description to generic backgrounds and to relativistic species. Indeed, for subrelativistic velocities and massive species, such that $p^\mu = mv^\mu = m(1, \boldsymbol{v})$ in flat Minkowski space-time, we recover Boltzmann equation,

$$m\partial_t f_s + mv^i \partial_i f_s + F_i \partial_{v_i} f_s = m\Gamma(f_s)\,. \tag{33}$$

Collisionless Boltzmann equation can be understood as the conservation of f along particle trajectories,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x^\mu(\lambda), p^\mu(\lambda)) = 0\,, \tag{34}$$

in analogy to the total derivative $D_t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla} + \boldsymbol{a}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}$. This equation is required to be satisfied in any reference frame and hence it directly provides the covariant formulation of Boltzmann equation.

4 Multi-fluid equations

In this section, we will take moments of the Vlasov equation by considering two-fluids, one composed by ions and one composed by electrons. This description is useful to understand the basic concepts of MHD and their range of applicability and can be generalized to a fluid composed by n multiple species.

The system of two-fluid equations still maintain a very rich phenomenology of plasma physics. With this system one can study Langmuir waves, ion-acoustic and ion plasma waves, electromagnetic waves (modified from vacuum), upper hybrid waves, electrostatic ion waves, lower hybrid waves, electron-cyclotron waves, whistler waves, R and L waves (responsible for Faraday rotation), Alfvén waves, fast magnetosonic waves, electrostatic drift waves, streaming instabilities, parametric instabilities. When we take the single-fluid MHD approach, some of these effects will not be described, as only the combined motion of the plasma is described.

We will then consider the Maxwell distribution function in equilibrium to study perfect fluids and will consider small deviations. Note that for plasmas, two-stream instabilities occur when a cold plasma has a relative velocity with respect to another plasma component, and make non-Maxwellian distributions to become Maxwellian, showing that Maxwellian distributions seem to appear even in the absence of abundant collisions.

4.1 Collisional invariants and fluid equations

In general, a collisional invariant ψ, satisfies the following condition (we omit subscript s for simplicity)

$$\int \Gamma(f)\,\psi\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{35}$$

Hence, if we multiply Eq. (27) by ψ and integrate over $\boldsymbol{v}$, we find

$$\int \psi\,\partial_t f\,\mathrm{d}^3\boldsymbol{v} + \int \psi\boldsymbol{v}\cdot\boldsymbol{\nabla} f\,\mathrm{d}^3\boldsymbol{u} + \frac{1}{m}\int \psi\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{36}$$

The first integral becomes

$$\partial_t \int \psi f\,\mathrm{d}^3\boldsymbol{v} = \partial_t\big(\langle\psi\rangle n\big) = \partial_t\langle\psi n\rangle, \tag{37}$$

where we note that ψ does not depend on t as it is a collisional invariant, and n commutes with $\langle\,\rangle$ as it does not depend on $\boldsymbol{v}$. The second integral is

$$\int \boldsymbol{\nabla}\cdot\big(\psi\boldsymbol{v}f\big)\,\mathrm{d}^3\boldsymbol{v} - \int f\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}^3\boldsymbol{v} = \boldsymbol{\nabla}\cdot\langle n\psi\boldsymbol{v}\rangle - n\langle\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\rangle, \tag{38}$$

and the third integral is

$$\begin{aligned}\int \psi\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f\,\mathrm{d}^3\boldsymbol{v} &= \int \boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\big(\psi\boldsymbol{F}f\big)\,\mathrm{d}^3\boldsymbol{v} - \int f\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\big(\psi\boldsymbol{F}\big)\,\mathrm{d}^3\boldsymbol{v} = -n\langle\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot(\psi\boldsymbol{F}\rangle\\ &= -\,n\langle\psi\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\boldsymbol{F}\rangle - n\langle\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\psi\rangle,\end{aligned} \tag{39}$$

where the first term is zero if one assures that f decays to zero when $\boldsymbol{v}\to\infty$. Hence, the equation found from each collisional invariant from Boltzmann equation is

$$\boxed{\partial_t\langle n\psi\rangle + \boldsymbol{\nabla}\cdot\langle n\psi\boldsymbol{v}\rangle - \langle n\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\rangle - \frac{n}{m}\langle\psi\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\boldsymbol{F}\rangle - \frac{n}{m}\langle\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\psi\rangle = 0\ .} \tag{40}$$

If we assume that the force $\boldsymbol{F}$ is independent of $\boldsymbol{v}$, then

$$\boxed{\partial_t\langle n\psi\rangle + \boldsymbol{\nabla}\cdot\langle n\psi\boldsymbol{v}\rangle - \langle n\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\rangle - \frac{n}{m}\langle\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\psi\rangle = 0\ .} \tag{41}$$

Note that Lorentz force is not independent of $\boldsymbol{v}$ and, hence, one needs to keep this term in the equations. We will omit this result and directly use the resulting fluid equation below. We define the mass density $\rho_m = nm$, such that its evolution is found from Eq. (40) when $\psi = m$,

$$\boxed{\partial_t \rho_m + \boldsymbol{\nabla} \cdot \big(\rho_m \boldsymbol{u}\big) = 0 \ ,} \tag{42}$$

which corresponds to the conservation of mass density, or the zero-th moment of the Boltzmann equation. Note that $\boldsymbol{u} = \langle \boldsymbol{v} \rangle$ is the bulk fluid velocity,

$$\boldsymbol{u}(t, \boldsymbol{x}) = \frac{1}{n(t, \boldsymbol{x})} \int \boldsymbol{v} f(t, \boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}^3 \boldsymbol{v} \,, \tag{43}$$

and should not be confused with the velocity of the particles. It corresponds to the average velocity of all the particles contained in a local fluid cell at $\boldsymbol{x}$. Deviations with respect to the bulk velocity are described by the particular velocity $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$, and moments of the particular velocity will be important fluid variables. For example, the pressure and heat flux tensors are the second and third-order moments of the peculiar velocity,

$$P_{ij} = mn \langle w_i w_j \rangle = mn \langle (v_i - u_i)(v_j - u_j) \rangle = mn \langle v_i v_j \rangle - mn u_i u_j \ , \tag{44}$$

$$\begin{aligned} Q_{ijl} &= \frac{1}{2} mn \langle w_i w_j w_l \rangle = \frac{1}{2} mn [\langle (v_i - u_i)(v_j - u_j)(v_l - u_l) \rangle] \\ &= \frac{1}{2} mn [\langle v_i v_j v_l \rangle - u_j \langle v_i v_l \rangle - u_l \langle v_i v_j \rangle - u_i \langle v_j v_l \rangle + 2 u_i u_j u_l] \,. \end{aligned} \tag{45}$$

Two well known fluid variables will appear in the following, the specific internal energy ε and the heat flux $\boldsymbol{q}$,

$$\varepsilon = \tfrac{1}{2} P_{ii}/(mn) = \tfrac{1}{2} \langle \boldsymbol{w}^2 \rangle \,, \qquad q_i = Q_{ijj} = \langle w_i \, \boldsymbol{w}^2 \rangle \,. \tag{46}$$

Incorporating the pressure tensor, the invariance of the momentum flux $\psi = m\boldsymbol{v}$ yields the first moment of the Boltzmann equation,

$$\boxed{\partial_t \big(\rho_m \boldsymbol{u}\big) + \boldsymbol{\nabla} \cdot \big(\rho_m \boldsymbol{u} \boldsymbol{u}\big) + \boldsymbol{\nabla} \cdot \boldsymbol{P} - \frac{\rho_m}{m} \boldsymbol{F} = 0 \,.} \tag{47}$$

The second moment of the Boltzmann equation is found using the kinetic energy of the particles $\psi = \frac{1}{2} m (\boldsymbol{v} - \boldsymbol{u})^2 = \frac{1}{2} m \boldsymbol{w}^2$, where $\boldsymbol{w}$ is the peculiar velocity. Then,

$$\boxed{\partial_t \big(\rho_m \varepsilon\big) + \boldsymbol{\nabla} \cdot \big(\rho_m \varepsilon \boldsymbol{u}\big) + \boldsymbol{\nabla} \cdot \boldsymbol{q} + P_{ij} S^{ij} = 0 \,,} \tag{48}$$

where S^{ij} is the rate-of-strain

$$S^{ij} = \tfrac{1}{2} \big(\partial^i u^j + \partial^j u^i\big) \,. \tag{49}$$

A similar dynamic equation is found for the pressure tensor P_{ij} if we take the second moment of Boltzmann equation, corresponding to the equation multiplied by $w_i w_j$ and integrated over $\boldsymbol{v}$.

In general, a particularity of this system can be observed: *any equation involving a time evolution of the rank-n tensor with n moments of the velocity includes the $n+1$ moment.* For example, the mass conservation equation for the zero-moment ρ_m includes the $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ term, the equation for the bulk velocity $\boldsymbol{u}$ includes the $\boldsymbol{\nabla}\cdot\boldsymbol{P}$ term, the dynamic equation for P_{ij} includes $\boldsymbol{\nabla}\cdot\boldsymbol{Q}$ and so on. This corresponds to the ***hierarchy problem*** (analogous to the BBGKY) and arises by the attempt to approximate a full seven-dimensional $f(t,\boldsymbol{x},\boldsymbol{v})$ function by a finite number of 4-dimensional fluid fields. We will require in general to give a prescription for some of the moments in terms of the lower moments that allows to close the system. For example, the equation of state of a fluid usually describes the pressure tensor P_{ij} as a function of ρ_m and ε and the flux tensor Q_{ijl} as a function of the internal energy ε.

In plasma physics, we will provide a fluid equation for each of the different species that compose the plasma. Mass, momentum, and energy conservation equations are

$$\partial_t n_s + \boldsymbol{\nabla}\cdot(n_s\boldsymbol{u}_s) = 0\,, \tag{50}$$

$$\partial_t(n_s\boldsymbol{u}_s) + \boldsymbol{\nabla}\cdot(n_s\boldsymbol{u}_s\boldsymbol{u}_s) = -\,\boldsymbol{\nabla}\cdot\boldsymbol{P}_s + \frac{q_s n_s}{m_s}\big(\boldsymbol{E} + \boldsymbol{u}_s\times\boldsymbol{B}\big)\,, \tag{51}$$

$$\partial_t(n_s\varepsilon_s) + \boldsymbol{\nabla}\cdot(n_s\varepsilon_s\boldsymbol{u}_s) = -\,\boldsymbol{\nabla}\cdot\boldsymbol{q}_s - P_{s,ij}S_s^{ij}\,. \tag{52}$$

Then, these equations can be solved together with Maxwell equations,

$$\partial_t\boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}\,,\qquad \partial_t\boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{J}\,,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_e\,,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \tag{53}$$

where the charge and current densities are

$$\rho_e = \sum_s q_s n_s\,,\qquad \boldsymbol{J} = \sum_s q_s n_s \boldsymbol{u}_s\,. \tag{54}$$

This system of equations is, however, not closed, as mentioned above so it can only be solved after we provide an appropriate equation of state to solve the system. For some applications, this system can be expanded to higher orders, as long as the hierarchical system is closed in a justified manner, which will depend on the distribution function of the fluid.

The corresponding covariant formulation of the mass and energy conservation of the fluid can be found taking moments of the relativistic Boltzmann equation Eq. (32). First, we define the first and second moments of the distribution function

$$J^\mu_{s,\mathrm{m}} = m\int p^\mu f_s \frac{\mathrm{d}^3p}{p^0}\,,\qquad T^{\mu\nu}_s = \int p^\mu p^\nu f_s \frac{\mathrm{d}^3p}{p^0}\,, \tag{55}$$

where J^μ_m is the four-mass current and $T^{\mu\nu}$ the stress-energy tensor. Since p^μ and $p^\mu p^\nu$ are collisional invariants, it directly follows

$$D_\mu J^\mu_{s,\mathrm{m}} = 0\,,\qquad D_\mu T^{\mu\nu}_s = \langle f^\nu\rangle\,, \tag{56}$$

where f^ν is the four-force density applied in the fluid, its temporal component representing energy dissipation and the spatial components the Newtonian force. Since the conservation of the stress-energy tensor is also a consequence of the Bianchi identities, one can find that any external forces applied to the fluid correspond to the divergence of the components of the stress-energy tensor that do not belong to the fluid. For example, forces due to electromagnetic forces (Lorentz force) corresponds to the covariant divergence of the stress-energy tensor associated to electromagnetic fields,

$$f^{\nu}_{\mathrm{Lor}} = -D_{\mu} T^{\mu\nu}_{\mathrm{EM}} \,. \tag{57}$$

These equations correspond to the conservation of mass, and the conservation of stress-energy. In the subrelativistic limit, for massive particles, the four-mass current is

$$J^0_{\mathrm{m}} = \rho_m \,, \qquad J^i_{\mathrm{m}} = \rho_m u^i \,, \tag{58}$$

and Eq. (50) for mass conservation is recovered. To obtain the components of the stress-energy tensor, we first expand up to second order the four-momentum, such that $p^0 = m\gamma \simeq m(1 + \frac{1}{2}v^2)$ and $p^i = m\gamma v^i \simeq m(1 + \frac{1}{2}v^2)v^i$, where γ is the Lorentz factor of the particle. Then, the stress-energy components become

$$T^{00} = m \int (1 + \tfrac{1}{2}v^2) f \, \mathrm{d}^3\boldsymbol{v} = \rho_m (1 + \varepsilon + \tfrac{1}{2}u^2) \,, \tag{59}$$

which corresponds to the total energy density (rest-mass, internal, and kinetic energies),

$$T^{0i} = m \int (1 + \tfrac{1}{2}v^2) v^i f \, \mathrm{d}^3\boldsymbol{v} = \rho_m (1 + \varepsilon + \tfrac{1}{2}u^2) u^i + \tfrac{1}{2}\rho_m q^i + u_j P^{ij} \,, \tag{60}$$

corresponding to the energy density fluxes, and

$$T^{ij} = m \int (1 + \tfrac{1}{2}v^2) v^i v^j f \, \mathrm{d}^3\boldsymbol{v} = \rho_m u^i u^j + P^{ij} + \frac{1}{2}\rho_m \langle v^2 v^i v^j \rangle \,, \tag{61}$$

where the last term involves higher-order moments. Incorporating the stress-energy components in the conservation law of Eq. (56), the energy and momentum conservation equations in a flat Minkowski space-time Eqs. (51) and (52) are recovered (as an exercise, show that combining the conservation laws, the subrelativistic fluid equations are recovered).

4.2 Maxwell-Boltzmann equilibrium distribution

The equilibrium distribution $f_0(\boldsymbol{v})$ corresponds to an asymptotic stationary solution of the Boltzmann equation. The H-theorem defines an H function that satisfies the following relations

$$\partial_t f_0(\boldsymbol{v}) = \Gamma(f) = 0 \Leftrightarrow f_1' f_2' - f_1 f_2 = 0 \Leftrightarrow \frac{\mathrm{d}H}{\mathrm{d}t} = 0 \,, \tag{62}$$

where $f_1'f_2' - f_1f_2$ is the integrand of the collision term in Eq. (28). The function H is

$$H(t) = \int f(t, \boldsymbol{v}) \ln\big[f(t, \boldsymbol{v})\big] \, \mathrm{d}^3\boldsymbol{v} \,, \tag{63}$$

and can be used to define the entropy of the fluid in a volume V,

$$S = -k_\mathrm{B} V H(t) = -k_\mathrm{B} V \int f(t, \boldsymbol{x}, \boldsymbol{v}) \ln\big[f(t, \boldsymbol{x}, \boldsymbol{v})\big] \, \mathrm{d}^3\boldsymbol{v}, \tag{64}$$

where k_B is the Boltzmann constant. Since $\mathrm{d}H/\,\mathrm{d}t \leq 0$, then $\mathrm{d}S/\,\mathrm{d}t \geq 0$.

The condition in the H-theorem is

$$f_1'f_2' = f_1f_2 \Rightarrow \ln f_1' + \ln f_2' = \ln f_1 + \ln f_2, \tag{65}$$

showing that it holds when $\ln f$ is a collisional invariant. Therefore, $\ln f_0$ is expressed as the superposition of N conserved quantities χ

$$\ln f_0(\boldsymbol{v}) = \sum_{i=1}^{N} \chi_i(\boldsymbol{v}) \ . \tag{66}$$

For a simple classical monatomic fluid, the conserved quantities are the kinetic energy, the total momentum, and the particle number, such that

$$\ln f_0(\boldsymbol{v}) = -A\boldsymbol{w}^2 + \ln C \ . \tag{67}$$

The equilibrium distribution function for a single component system (e.g., monatomic fluid of identical particles), for which the effects of the external forces are negligible, is the Maxwell-Boltzmann distribution. Using Eq. (67), we can find

$$n = \int f_0(\boldsymbol{v}) \, \mathrm{d}^3\boldsymbol{v} = C \int e^{-A\boldsymbol{w}^2} \, \mathrm{d}^3\boldsymbol{w} = 4\pi C \int_0^\infty w^2 e^{-Aw^2} \, \mathrm{d}w = C \left(\frac{\pi}{A}\right)^{3/2} , \tag{68}$$

where we have used spherical coordiantes to express the integral over $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ as an integral over the peculiar speed w. The specific internal energy of the fluid allows us to compute the constant A,

$$\varepsilon = \frac{C}{2n} \int \boldsymbol{w}^2 e^{-A\boldsymbol{w}^2} \, \mathrm{d}^3\boldsymbol{w} = \frac{2\pi C}{n} \int_0^\infty w^4 e^{-Aw^2} \, \mathrm{d}w = \frac{3}{4} \frac{\pi^{3/2}}{A^{5/2}} \frac{C}{n} = \frac{3}{4A} \, . \tag{69}$$

Defining the temperature for a classical monatomic fluid, such that $\varepsilon = \frac{3}{2} k_\mathrm{B} T/m$, we find

$$A = \frac{m}{2k_B T} \, , \qquad C = n \left(\frac{m}{2\pi k_B T}\right)^{3/2} . \tag{70}$$

Hence, the Maxwell-Boltzmann distribution is

$$\boxed{f_0(\boldsymbol{v}) = n \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{\frac{3}{2}} \exp\left(-\frac{m|\boldsymbol{v} - \boldsymbol{u}|^2}{2k_\mathrm{B} T}\right) \ .} \tag{71}$$

The root mean square velocity is

$$\langle \boldsymbol{v}^2 \rangle = \int w^2 f_0(\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v} + \boldsymbol{u}^2 = \frac{3k_\mathrm{B}T}{m} + \boldsymbol{u}^2 \Rightarrow \boxed{T = \frac{m}{3k_\mathrm{B}}\left[\langle \boldsymbol{v}^2 \rangle - \boldsymbol{u}^2\right] = \frac{m\langle w^2 \rangle}{3k_\mathrm{B}}\ .} \tag{72}$$

A procedure analogous to the one followed to obtain the Maxwell-Boltzmann distribution yields the following equilibrium distribution in the relativistic counterpart

$$f_0 = \frac{g_s}{h_\mathrm{P}^3} \frac{1}{\exp(-\mu/T - p^\mu u_\mu/T)}\ , \tag{73}$$

where μ is the chemical potential and g_s is the degeneracy factor, $g_s = 2s + 1$ for $m \neq 0$ and $2s$ for $m = 0$ for s internal spin degrees of freedom. This is the Maxwell-Jütner distribution and applies to non-degenerate fluid. Considering degenerate fluids (allowing to include fermions and bosons), one can find

$$f_0 = \frac{g_s}{h_\mathrm{P}^3} \frac{1}{\exp[(E - \mu)/T] \pm 1}\ , \tag{74}$$

with $+1$ for fermions (Fermi-Dirac distribution) and -1 for bosons (Bose-Einstein distribution).

4.3 Perfect fluids

The zero-th order approximation in hydrodynamics corresponds to the assumption that the distribution function is that found in local thermal equilibrium (LTE), i.e., the Maxwell-Boltzmann distribution in the subrelativistic limit; or one of the equilibrium distribution functions in the relativistic limit (Maxwell-Jüttner, Bose-Einstein, or Fermi-Dirac). This corresponds to the perfect fluid description.

The perfect fluid description holds when the collisions in the system are frequent, such that they drive the fluid parcels to LTE in a time scale l_mfp/v that is shorter than the time scales characterizing the fluid, being v the typical particle velocity and l_mfp the mean-free path. This corresponds to the limit of small Knudsen number, $\mathrm{Kn} = l_\mathrm{mfp}/L \ll 1$, i.e., small mean-free path of the fluid particles compared to a characteristic length scale L of the fluid fields.

In a perfect fluid, we can then solve the fluid equations found from taking the moments of the Boltzmann equation [see Eqs. (42), (47) and (48)], where the pressure tensor P_{ij} and the heat flux q_i can be computed using the Maxwell-Boltzmann distribution,

$$\begin{aligned} P_{ij} &= m\langle w_i w_j \rangle = m \int w_i w_j f_0(\boldsymbol{w})\, \mathrm{d}^3\boldsymbol{w} = 4\pi m C \int w^2 w_i w_j e^{-Aw^2}\, \mathrm{d}w \\ &= \frac{4\pi}{3} m C \delta_{ij} \int_0^\infty w^4 e^{-Aw^2}\, \mathrm{d}w = \frac{2}{3} \rho_m\, \varepsilon\, \delta_{ij} = p\delta_{ij}\ , \end{aligned} \tag{75}$$

where we have defined the isotropic pressure p. The pressure tensor becomes isotropic as a consequence of the symmetry of Maxwell distribution around w^2. This is the case for perfect fluids but we can have anisotropic pressure tensors in real fluids (e.g., when incorporating viscosity). For a classical monatomic fluid, we recover the pressure

$$p = nk_B T\,. \tag{76}$$

Also as a consequence of the symmetry of the Maxwell distribution, it is automatically satisfied that heat fluxes are zero,

$$\boldsymbol{q} = \tfrac{1}{2}\rho_m \langle w^2 \boldsymbol{w}\rangle = 0\,. \tag{77}$$

Going back to the subrelativistic limit, the system of equations in LTE are then the continuity equation (mass conservation),

$$\partial_t \rho_m + \boldsymbol{\nabla}\cdot(\rho_m \boldsymbol{u}) = 0\ , \tag{78}$$

the Euler equation,

$$\partial_t \boldsymbol{u} + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\boldsymbol{u} + \frac{1}{\rho_m}\boldsymbol{\nabla} p - \frac{\boldsymbol{F}}{m} = 0\ , \tag{79}$$

and the conservation of energy (note that $P_{ij}S^{ij} = p\boldsymbol{\nabla}\cdot\boldsymbol{u}$)

$$\partial_t \varepsilon + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\varepsilon + \frac{p}{\rho_m}\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ . \tag{80}$$

We can find an equation for $\frac{1}{2}u^2$ by taking the dot product of Euler equation with $\boldsymbol{u}$,

$$\tfrac{1}{2}\partial_t u^2 + \boldsymbol{u}\cdot\big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\boldsymbol{u} + \frac{1}{\rho_m}\boldsymbol{u}\cdot\boldsymbol{\nabla} p - \frac{\boldsymbol{u}\cdot\boldsymbol{F}}{m} = 0 \tag{81}$$

where we can use $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla} u^2$ and divide by u^2. We can also divide the continuity equation by ρ_m. Then, we find

$$\tfrac{1}{2}\partial_t \ln u^2 + \tfrac{1}{2}\big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\ln u^2 + \frac{1}{\rho_m u^2}\boldsymbol{u}\cdot\boldsymbol{\nabla} p - \frac{\boldsymbol{u}\cdot\boldsymbol{F}}{m u^2} = 0\,, \tag{82}$$

$$\partial_t \ln \rho_m + \boldsymbol{u}\cdot\boldsymbol{\nabla}\ln\rho_m + \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ . \tag{83}$$

Expressing them via the logarithm allows us to directly sum both equations to find a dynamical equation for the subrelativistic kinetic energy $\frac{1}{2}\rho_m u^2$,

$$\partial_t\Big(\tfrac{1}{2}\rho_m u^2\Big) + \boldsymbol{\nabla}\cdot\Big(\tfrac{1}{2}\rho_m u^2 \boldsymbol{u}\Big) + \boldsymbol{u}\cdot\boldsymbol{\nabla} p - \rho_m\frac{\boldsymbol{u}\cdot\boldsymbol{F}}{m} = 0\ . \tag{84}$$

Similarly, we divide the conservation of energy equation by ε and add to the continuity equation to find

$$\partial_t\big(\rho_m\varepsilon\big) + \boldsymbol{\nabla}\cdot\big(\rho_m\varepsilon\boldsymbol{u}\big) + p\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ . \tag{85}$$

We can combine the equations for the kinetic and internal energies,

$$\partial_t\Big(\tfrac{1}{2}u^2\rho_m + \varepsilon\rho_m\Big) + \boldsymbol{\nabla}\cdot\Big[\Big(\tfrac{1}{2}u^2\rho_m + \varepsilon\rho_m + p\Big)\boldsymbol{u}\Big] = \frac{\rho_m}{m}\boldsymbol{u}\cdot\boldsymbol{F}\,. \tag{86}$$

The primordial plasma of the early Universe, after the period of reheating, is dominated by massless radiation particles. It can be shown that independently of their distribution (i.e., if they are fermions, bosons, or fluid ensembles), the energy density and the isotropic pressure are

$$p = \frac{1}{3}\rho\,, \tag{87}$$

such that the sound speed is

$$c_{\rm s}^2 = \frac{\partial p}{\partial \rho}\bigg|_s = \frac{1}{3}\,. \tag{88}$$

For massless particles, then $J^\mu_{\rm m} = 0$ and the conservation laws reduce to

$$D_\mu T^{\mu\nu} = 0\,, \tag{89}$$

which now becomes a closed system (energy and momentum conservation) as $P_{ij} = p\delta_{ij} = c_{\rm s}^2\rho\delta_{ij}$ and $q_i = 0$.

Note that the covariant formulation of the relativistic stress-energy tensor for the perfect fluid becomes

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu + pg^{\mu\nu}\,, \tag{90}$$

as a consequence of the symmetry of the equilibrium distribution functions, such that the pressure tensor becomes $pg^{\mu\nu}$, i.e., isotropic in isotropic metric tensors like Minkowski, and heat fluxes are zero. This tensor is also the general tensor that can describe a homogeneous and isotropic Universe and hence, it is commonly used to express any component of the primordial Universe.

4.4 Imperfect fluids

The zero-th order approximation corresponds to the zero-order expansion around the equilibrium distribution function and does not account for dissipative effects that lead to the entropy production. To go beyond this assumption, the distribution function is expanded using the Knudsen number as the perturbative parameter, following Chapman-Enskog theory. This allows to describe imperfect fluids in the first-order hydrodynamics approximation, leading to Navier-Stokes equations describing viscosity, and to heat fluxes, described by Fourier's law of conductivity. In particular, each of these terms appear when expanding around Knudsen numbers based on the length scale associated to the shear stress and the temperature gradients, respectively.

The distribution function is expanded as

$$f(t,\boldsymbol{x},\boldsymbol{v}) = f_0(t,\boldsymbol{x},\boldsymbol{v}) + f_1(t,\boldsymbol{x},\boldsymbol{v}) + \mathcal{O}(\mathrm{Kn}^2)\,. \tag{91}$$

In the subrelativistic limit, the additional contribution to the stress-energy tensor due to first-order imperfect fluids is described by the deviatoric viscous stress tensor Π_{ij}, such that $P_{ij} = p\delta_{ij} - \Pi_{ij}$

$$\Pi^{ij} = 2\,\nu\,\sigma^{ij} + \zeta\,\theta\,\delta^{ij}\,, \tag{92}$$

where is the rate-of-strain tensor, $\theta = S^i{}_i = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ is the fluid expansion scalar, and $\sigma^{ij} = S^{ij} - \frac{1}{3}\theta\,\delta^{ij}$ the traceless rate-of-strain tensor. A kinematic approach to describe the anisotropic stress tensor, T_{ij}, as the more general 2-rank tensor to describe linear constitutive relations (i.e., Newtonian fluids) also leads to Eq. (92) and corresponds to the original derivation of the Navier-Stokes equations. The divergence of the deviatoric stress tensor corresponds to the viscous force,

$$f^i_{\mathrm{visc}} = \partial_j \Pi^{ij}\,. \tag{93}$$

A particular choice of the viscous coefficients is such that the stress-energy tensor trace is zero. Under this condition, known as the Stokes assumption, the thermodynamic and mechanical pressure are equivalent. Hence, the Stokes assumption is satisfied when the bulk viscosity vanishes, $\zeta = 0$, as $\sigma^i{}_i = 0$ by construction.

Similarly, Chapman-Enskog theory also allows to obtain Fourier's law of conductivity, where the heat flux is found to be

$$\boldsymbol{q} = -\kappa\boldsymbol{\nabla}T\,. \tag{94}$$

The fluid equations (let us omit the Lorentz force for simplicity) in the first-order approximation (following Chapman-Enskog theory) become the following. The continuity equation stays the same. The Euler equation becomes the Navier-Stokes equation, including viscous effects from the shear stress tensor,

$$\boxed{\partial_t\boldsymbol{u} + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\boldsymbol{u} = -\frac{1}{\rho_m}\boldsymbol{\nabla}\Big(p - \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\Big) + \frac{2}{\rho_m}\boldsymbol{\nabla}\cdot\big(\nu\boldsymbol{S}\big)\,,} \tag{95}$$

where $\lambda = \zeta - \frac{2}{3}\nu$ is the dilatational viscous coefficient. If the viscous coefficients are homogeneous, we get

$$\partial_t\boldsymbol{u} + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\boldsymbol{u} = -\frac{1}{\rho_m}\boldsymbol{\nabla}p + \frac{\nu}{\rho_m}\boldsymbol{\nabla}^2\boldsymbol{u} + \frac{1}{\rho_m}\big(\zeta + \tfrac{1}{3}\nu\big)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u}\,. \tag{96}$$

Under the Stokes assumption, Navier-Stokes equation becomes

$$\partial_t\boldsymbol{u} + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\boldsymbol{u} = -\frac{1}{\rho_m}\boldsymbol{\nabla}p + \frac{\nu}{\rho_m}\boldsymbol{\nabla}^2\boldsymbol{u} + \frac{\nu}{3\rho_m}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u}\,. \tag{97}$$

The heat conduction equation, where the thermal conductivity characterizes the heat flux $q_i = -\kappa\partial_i T$, is included in the conservation of energy,

$$\boxed{\partial_t\varepsilon + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\varepsilon = \frac{1}{\rho_m}\boldsymbol{\nabla}\cdot\big(\kappa\boldsymbol{\nabla}T\big) - \frac{p}{\rho_m}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{\lambda}{\rho_m}\big(\boldsymbol{\nabla}\cdot\boldsymbol{u}\big)^2 + \frac{2\nu}{\rho_m}S_{ij}S_{ij}\,,} \tag{98}$$

where $\Pi_{ij}S_{ij} = \lambda\big(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2 + 2\nu S_{ij}S_{ij} \geq 0$ represents the irreversible conversion of mechanical to thermal energy through the action of viscosity. The heat-diffusion equation is found when $\boldsymbol{u} = 0$, for a constant κ,

$$\partial_t\varepsilon = \frac{3k_{\mathrm{B}}}{2m}\partial_t T = \frac{\kappa}{\rho_m}\boldsymbol{\nabla}^2 T \Rightarrow \partial_t T = \frac{2m\kappa}{3k_{\mathrm{B}}\rho_m}\boldsymbol{\nabla}^2 T\ . \tag{99}$$

The specific entropy is defined by the H function [see Eq. (64)],

$$s = S/V = -k_{\mathrm{B}}\int f(t,\boldsymbol{x},\boldsymbol{v})\ln\big[f(t,\boldsymbol{x},\boldsymbol{v})\big]\,\mathrm{d}^3\boldsymbol{v}, \tag{100}$$

which, for the Maxwell-Boltzmann distribution is

$$s = \frac{3}{2}nk_{\mathrm{B}}\ln\big(pV^{5/3}\big) + C, \tag{101}$$

where C is a constant. The second law of thermodynamics can be derived from this definition of the entropy, using the H integral, where we ignore variations of the chemical potential μ that would enter in the second law,

$$\mathrm{d}s = \frac{3}{2}nk_{\mathrm{B}}\,\mathrm{d}\ln(pV^{5/3}) = \frac{3}{2}nk_{\mathrm{B}}\left(\frac{\mathrm{d}p}{p} + \frac{5}{3}\frac{\mathrm{d}V}{V}\right). \tag{102}$$

We now take into account that, for a fluid in LTE, $p = nk_{\mathrm{B}}T = \frac{2}{3}\rho\varepsilon$, and $V = m/\rho$. Hence,

$$T\,\mathrm{d}s = \rho\varepsilon\left(\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}\varepsilon}{\varepsilon} - \frac{5}{3}\frac{\mathrm{d}\rho}{\rho}\right) = \rho\,\mathrm{d}\varepsilon - p\,\mathrm{d}\rho/\rho \Rightarrow \mathrm{d}\varepsilon = T\,\mathrm{d}s/\rho - p\,\mathrm{d}(1/\rho) = T\,\mathrm{d}s - p\,\mathrm{d}V\ . \tag{103}$$

From the second law of thermodynamics, we can find an evolution equation for the entropy, taking into account $\mathrm{d} = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$,

$$\begin{aligned}\partial_t s + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)s &= \partial_t\varepsilon + \big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\varepsilon - \frac{p}{\rho_m}\partial_t\rho_m - \frac{p}{\rho_m}\big(\boldsymbol{u}\cdot\boldsymbol{\nabla}\big)\rho_m \\ &= \frac{1}{T\rho_m}\Big[\boldsymbol{\nabla}\cdot\big(\kappa\boldsymbol{\nabla}T\big) + \Pi_{ij}S_{ij}\Big] \geq 0\ ,\end{aligned} \tag{104}$$

showing that the total variation of entropy is due to heat transfer and the viscous stresses. Hence, the entropy is completely conserved for a perfect fluid.

The covariant formulation of Navier-Stokes viscosity and Fourier's heat fluxes (first-order hydrodynamic theories) is known as the classical irreversible thermodynamics (CIT) approach. A well-known problem of the CIT approach is that it leads to fluid perturbations that are allowed to propagate at unbounded speeds, violating the postulates of special relativity. Causality violation is a consequence of the parabolic nature of the diffusion operators that describe Navier-Stokes viscosity in the momentum equation, $\nu\,\boldsymbol{\nabla}^2\boldsymbol{u}$, and Fourier's heat flux in the energy equation, $\kappa\,\boldsymbol{\nabla}^2 T$, being ν and κ the coefficients of shear viscosity and heat-flux conductivity, respectively.

Following the CIT approach, the deviatoric viscous stress tensor is $\Pi^{\mu\nu}$,

$$\Pi^{\mu\nu} = 2\nu\sigma^{\mu\nu} + \zeta\theta h^{\mu\nu}\,, \qquad \sigma^{\mu\nu} = S^{\mu\nu} - \frac{1}{3}\theta h^{\mu\nu}, \tag{105}$$

where $\theta = D_\mu U_\mu$ is the relativistic dilatational coefficient. $S^{\mu\nu}$ is the relativistic rate-of-strain tensor,

$$S^{\mu\nu} = \frac{1}{2}\left(h^{\alpha\nu} D_\alpha U^\mu + h^{\alpha\mu} D_\alpha U^\nu\right)\,, \tag{106}$$

where $h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ is the spatial projection tensor. We can rearrange Eq. (105) in the following way

$$\Pi^{\mu\nu} = 2\nu S^{\mu\nu} + \lambda\theta h^{\mu\nu}\,, \tag{107}$$

where $\lambda = \zeta - \frac{2}{3}\nu$. The term $\sigma^{\mu\nu}$ is traceless by construction,

$$\sigma^\mu{}_\mu = S^\mu{}_\mu - \tfrac{1}{3}\theta h^\mu{}_\mu = 0\ , \tag{108}$$

while the bulk viscosity yields the trace $\zeta\theta h^\mu{}_\mu = 3\zeta\theta$ to the deviatoric tensor,

$$\Pi^\mu{}_\mu = 3\,\zeta\,\theta\ . \tag{109}$$

However, this is not an appropriate theory for relativistic dissipative fluids and one needs to go beyond this approach. This was first done by Cattaneo, including relaxation times in the constitutive relations,

$$\tau_r\,\partial_\tau \boldsymbol{q} + \boldsymbol{q} = -\kappa\boldsymbol{\nabla}T\,, \tag{110}$$

$$\tau_\Pi\,\partial_\tau \Pi_2^{ij} + \Pi_2^{ij} = \Pi_1^{ij}\,, \tag{111}$$

leading to the fluid equations known as Maxwell-Cattaneo equations, later adapted in Israel-Stewart theories for relativistic fluid dynamics. In the following, for simplicity, we will only consider the subrelativistic Navier-Stokes description.

5 Single-fluid MHD equations

In previous section, we have considered first-order hydrodynamic theories separately for each species. When we want to include collisions, we need to recover the effect of $(\,\mathrm{d}f_s/\,\mathrm{d}t)_c$ in the right hand side of Vlasov equation and take into account also the effect due to collisions between different species. We will consider a fluid composed by two species: electrons and ions, as commonly considered in MHD.

We expect collisions to mostly have an impact on the velocity-dependent part of the distribution function as collisions can change significantly the velocity of the particles but not as much the particle's positions. Then, we can ignore the effect of collisions in the continuity equation (the number of particles in a small volume is almost unaffected). On the other hand, in the momentum equation, we get the additional collisional term

$$\boldsymbol{K}_s = m_s \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}(\,\mathrm{d}f_s/\,\mathrm{d}_t)_c\,. \tag{112}$$

We note that the momentum of a species cannot change by colliding with itself as a consequence of being a collisional invariant. However, when having two fluids, the momentum of electrons in a volume will be changed by collisions with the ions and we expect that the change in momentum of one species will have a corresponding effect on the other species, $\boldsymbol{K}_e = -\boldsymbol{K}_i$. This allows to simplify the two-fluid equations by combining them into a one-fluid model (MHD). The single fluid has the following mass density, charge density and velocity

$$\rho_m = m_e n_e + m_i n_i \approx m_i n_i \,, \quad \rho_e = q_e n_e + q_i n_i \approx e(n_i - n_e) \,, \tag{113}$$

$$\boldsymbol{u} = \frac{1}{\rho_m}(m_i n_i \boldsymbol{u}_i + m_e n_e \boldsymbol{u}_e) \,, \tag{114}$$

the following current density and total pressure

$$\boldsymbol{J} = q_i n_i \boldsymbol{u}_i + q_e n_e \boldsymbol{u}_e \,, \quad P = P_i + P_e \,. \tag{115}$$

We are now ready to derive the MHD equations. Let us start with the mass conservation law, found combining the continuity equation for both species,

$$\boxed{\partial_t \rho_m + \boldsymbol{\nabla} \cdot (\rho_m \boldsymbol{u}) = 0 \,.} \tag{116}$$

Multiplying the ion continuity equation by q_i and the electron continuity equation by q_e and adding we find the charge conservation law

$$\partial_t \rho_e + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \,, \tag{117}$$

which is also a direct consequence of Maxwell equations.

Combining the momentum equations for electrons and ions we find

$$\boxed{\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} P + \boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{F}/m + \Pi_{ij} \,,} \tag{118}$$

where we have reintroduced mass forces that can change the momentum (e.g., gravity) and the deviatoric stress tensor Π_{ij} due to the viscosity that was found following Navier-Stokes description (or computed in Chapman-Enskog theory) to take the form

$$\Pi_{ij} = 2\nu (S_{ij} - \tfrac{1}{3} \boldsymbol{\nabla} \cdot \boldsymbol{u} \delta_{ij}) \,, \tag{119}$$

assuming Stoke's hypothesis (zero bulk viscosity).

Finally, we can find a generalized Ohm's law by multiplying the momentum equation by q_s/m_s and adding the ion version to the electron version. Furthermore, we neglect quadratic terms and use $q_i = -q_e = e$, to find

$$\begin{aligned} \partial_t \boldsymbol{J} = &-\frac{e}{m_i} \boldsymbol{\nabla} P_i + \frac{e}{m_e} \boldsymbol{\nabla} P_e + \left(\frac{e^2 n_e}{m_e} + \frac{e^2 n_i}{m_i} \right) \boldsymbol{E} + \frac{e^2 n_e}{m_e} \boldsymbol{u}_e \times \boldsymbol{B} \\ &+ \frac{e^2 n_i}{m_i} \boldsymbol{u}_i \times \boldsymbol{B} + \left(\frac{e}{m_i} + \frac{e}{m_e} \right) \boldsymbol{K}_i \,. \end{aligned} \tag{120}$$

We can simplify this expression by taking $m_e \ll m_i$, assuming charge neutral plasmas $n_e \sim n_i$ (note that this does not imply $\boldsymbol{J} = 0$ for the MHD fluid), $P_i \sim P_i \sim \frac{1}{2}P$. Then,

$$\partial_t \boldsymbol{J} = \frac{e}{2m_e}\boldsymbol{\nabla} P + \frac{e^2\rho}{m_e m_i}\big(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\big) - \frac{e}{m_e c}\boldsymbol{J} \times \boldsymbol{B} + \frac{e}{m_e}\boldsymbol{K}_i\,. \tag{121}$$

We can already identify some known terms! The first term due to pressure gradients is the responsible of the Biermann battery, and it can usually be ignored for small pressure gradients. The Hall term corresponds to the $\boldsymbol{J} \times \boldsymbol{B}$ term and is usually ignored when compared to $\boldsymbol{u} \times \boldsymbol{B}$. Then, at very low frequencies, $\partial_t \boldsymbol{J}$ can also be ignored, and we are left with

$$\boldsymbol{K}_i = -\frac{e\rho}{m_i}(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})\,. \tag{122}$$

We can argue that the collision term corresponding to the change of momentum in ions due to electrons should be up to first order proportional to $\boldsymbol{K}_i = C_1(\boldsymbol{u}_i - \boldsymbol{u}_e) = -C_2 \rho \boldsymbol{J}$. Introducing this, we can recognize the Ohm's law if we set $C_2 = \rho e/(m_i \sigma)$,

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})\,. \tag{123}$$

By neglecting all the aforementioned terms, we find the resistive MHD equations, after we couple the fluid equations found to the Maxwell equations. In the limit of a collisionless fluid, then the conductivity becomes infinite, and Ohm's law yields $\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$, which is then the limit of ideal MHD, and no charges imbalances are allowed. The covariant formulation of Ohm's law becomes

$$J^\mu = \rho_e U^\mu + \sigma E^\mu\,. \tag{124}$$

In general, we will work with resistive MHD as diffusivity effects becomes very relevant in the early Universe description of magnetic fields as, for example, in MHD turbulent decay and in processes like magnetic reconnection and dynamos.

6 Induction equation

We now focus on the Maxwell equations, that determine the dynamical evolution of the magnetic field,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_e\,, \quad \partial_t \boldsymbol{E} = \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J}\,, \tag{125}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0\,, \quad \partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E}\,. \tag{126}$$

The electric field in Faraday's law of induction is determined by Ohm's law $\boldsymbol{J} = \sigma \boldsymbol{E}$ which generalizes to $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ for a fluid in motion. Introducing the magnetic diffusivity $\eta = 1/\sigma$, Ampère's law can be expressed as

$$\left(\frac{1}{\eta} + \partial_t\right)\boldsymbol{E} = -\left(\frac{\boldsymbol{u}}{\eta} - \boldsymbol{\nabla}\right) \times \boldsymbol{B}\,. \tag{127}$$

Comparing both terms multiplying the electric field, we can estimate that the displacement current is negligible when the time scale of the electric field variations exceeds the Faraday time $t_{\rm Faraday} = \eta/c^2$. This is the case in all the situations we will be considering in astrophysical plasmas as well as in the primordial plasma after the onset of radiation-domination when the Universe becomes extremely hot and dense. Under this approximation, we find

$$\nabla \times \boldsymbol{B} = \boldsymbol{J} = \frac{1}{\eta}(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})\,, \tag{128}$$

such that the electric field is

$$\boldsymbol{E} = \eta \boldsymbol{J} - \boldsymbol{u} \times \boldsymbol{B}\,. \tag{129}$$

We can then take the curl of Ampère's law to find the induction equation (under the large conductivity limit),

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times \big(\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J}\big) = \boldsymbol{\nabla} \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) + \eta \nabla^2 \boldsymbol{B}\,. \tag{130}$$

Using dimensional analysis, we can compare the non-linear induction term, which is of order $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \sim UB/L$ with the magnetic diffusivity term $\eta \nabla^2 \boldsymbol{B} \sim \eta B/L^2$. Hence, we can define the magnetic Reynolds number as the ratio of non-linear to diffusion terms

$${\rm Re}_m = \frac{UL}{\eta}\,, \tag{131}$$

in a similar way as the Reynolds number. This determines the scales at which the diffusivity becomes relevant. The ratio of Reynolds to magnetic Reynolds number is defined as the magnetic Prandtl number

$${\rm Pr}_m = \frac{\nu}{\eta}\,, \tag{132}$$

and determines the ratio of viscous to magnetic diffusion rate.

7 Summary of MHD equations in the subrelativistic limit

In addition to Maxwell equations, which reduce to the induction equation when we can neglect the displacement current,

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{B}\,, \qquad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}\,, \tag{133}$$

we need to solve for the fluid conservation equations, studied in previous lectures. First, we consider the continuity equation

$$\partial_t \rho_m + \boldsymbol{\nabla} \cdot (\rho_m \boldsymbol{u}) = 0\,, \tag{134}$$

and we include Navier-Stokes equation with the Lorentz force,

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\rho}\boldsymbol{J} \times \boldsymbol{B} + 2\nu \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \,, \tag{135}$$

where $\sigma_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}) - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla} \cdot \boldsymbol{u}$ is the traceless rate-of-strain, determining the viscous-stress tensor. The Lorentz force can be expressed as

$$\boldsymbol{J} \times \boldsymbol{B} = -\boldsymbol{\nabla} B^2 + \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B} \,, \tag{136}$$

such that B^2 acts as a pressure term, giving a total pressure $p^* = p + B^2$. The ratio among both contributions is the β parameter and characterizes the strength of a magnetic field in a plasma,

$$\beta = \frac{p}{v_{\mathrm{A}}^2} \,. \tag{137}$$

The conservation of internal energy also includes Ohmic dissipation,

$$\partial_t \rho + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho + p \boldsymbol{\nabla} \cdot \boldsymbol{u} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + 2\nu \boldsymbol{S} : \boldsymbol{\nabla} \boldsymbol{u} - \eta J^2 \,. \tag{138}$$

Hence, the viscosity and magnetic diffusivity dissipate kinetic energy from the system into heat.