CosmoLattice

— School 2025 —

CosmoLattice School 2025 (IBS, Korea), Sept 22-26

MHD in the early Universe and its lattice formulation

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Part I Antonino Salvino Midiri (University of Geneva, Switzerland)

Lattice formulation of perfect fluids in flat spacetime

Part II Kenneth Marschall (IFIC, Valencia)

Lattice formulation of non-perfect fluids coupled to gauge fields in a FLRW expanding background (with gravitational waves)

...Sub-Relativistic MHD in flat spacetime

The energy-momentum tensor for a perfect fluid dominated by sub-relativistic massive particles in flat spacetime is

$$T_{pf}^{\mu\nu} = (\rho_m + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$

ρ_m mass density

p pressure

u^μ 4-velocity

$\eta^{\mu\nu}$ Minkowski metric

[See Lecture 9]

$$\partial_\mu T_{pf}^{\mu\nu} = 0$$

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$$\partial_\mu T^{\mu\nu}_{pf} = 0$$

$\xrightarrow{\nu = 0}$

$$J^\mu = \rho_m u^\mu$$

$$\partial_\mu J^\mu = 0$$

Conservation of mass

[See Lecture 9]

$(p \ll \rho_m)$

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Conservation of mass

[See Lecture 9]

$\nu = i$:

$$\partial_0(\rho_m u^i) = -\partial_j[\rho_m u^i u^j + p\, \delta^{ij}]$$

Conservation of momentum

$(p \ll \rho_m)$

General Relativistic MHD

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Metric tensor	$g^{\mu\nu}$
Fluid 4-velocity	u^μ
Fluid density	ρ
Fluid pressure	p

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$$f^\nu_{visc} \equiv \partial_\mu \Pi^{\mu\nu}_{visc} \qquad f^\nu_{Lorentz} \equiv -\partial_\mu T^{\mu\nu}_{EM}$$

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$\gamma = (1 - \boldsymbol{u}^2)^{-1/2}$

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Let us first focus (for simplicity) on perfect fluids (no viscosity) without any interaction with electromagnetic fields

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We have 5 variables $(\mathbf{u}, \rho, p)$ but only 4 equations

How can we solve this system?

Minkowski metric $\eta^{\mu\nu}$ = diag(-1,1,1,1)

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How can we solve this system? → Constant equation of state (c_s^2) relating p to ρ

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How can we solve this system? → Constant equation of state (c_s^2) relating p to ρ

We now have 4 variables $(\mathbf{u}, \rho)$ and 4 equations → closed system

Perfect fluids in flat spacetime

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$

We need to choose the 4 variables to solve for $\rightarrow$ not necessarily ρ and $\mathbf{u}$

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$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \qquad (i = 1,2,3) \end{cases}$$

T^{00}, T^{0i} seem to be a natural choice

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T^{00}, T^{0i} seem to be a natural choice $\longrightarrow$ We need to relate T^{ji} to them in order to close the system

$$T^{00} = \rho(1 + c_s^2)\gamma^2 - c_s^2\, \rho$$

$$T^{0i} = \rho(1 + c_s^2)\gamma^2 u^i$$

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$$r^2 = T^{0i} T^{0i} / (T^{00})^2$$

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$$\longrightarrow \quad r^2 = \frac{T^{0i}T^{0i}}{[T^{00}]^2} = \frac{(1 + c_s^2)^2 \gamma^4 u^2}{[(1 + c_s^2)\gamma^2 - c_s^2]^2}$$

$$u^2 = 1 - 1/\gamma^2$$

$$r^2 = T^{0i}T^{0i}/(T^{00})^2$$

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$$\longrightarrow \gamma^2 = \frac{1}{2(1 - r^2)}\left[1 - \frac{2r^2 c_s^2}{1 + c_s^2} + \sqrt{1 - \frac{4r^2 c_s^2}{(1 + c_s^2)^2}}\right]$$

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Perfect fluids in flat spacetime

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$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

CONSERVATION FORM

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

How do we solve them in the lattice?

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How do we solve them in the lattice?

After discretizing the derivatives we get equations of the form

$$\partial_0 X^\mu = \mathcal{K}^\mu[X^\nu]$$

← The RHS is a function of the fields themselves

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Natural algorithm for timestepping → **explicit Runge-Kutta** [See Lecture 3]

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$\partial_0 T^{00} = -\,\partial_j T^{j0} \longrightarrow \mathcal{K}^0[T^{0\mu}]$

$\partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \longrightarrow \mathcal{K}^i[T^{0\mu}]$

space discretization

$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_{\mathrm{j}} \mathrm{T}^{\mathrm{j}0}$$

$$\mathcal{K}^i[T^{0\mu}] \equiv \nabla_{\mathrm{j}} \mathrm{T}^{\mathrm{ji}}$$

Fluid dynamics in the **conservation form**

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$$\mathcal{K}^0[T^{0\mu}] \equiv \nabla_\mathrm{j} \mathrm{T}^{\mathrm{j}0}$$

For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

$$\mathcal{K}^i[T^{0\mu}] \equiv \nabla_\mathrm{j} \mathrm{T}^{\mathrm{ji}}$$

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FORWARD DERIVATIVE

$$\nabla_i^+ f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x})}{\delta x} \to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x)$$

$\hat{\boldsymbol{\imath}}$ unit vectors in the three spatial directions

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases} \quad \begin{matrix} \longrightarrow \mathcal{K}^0[T^{0\mu}] \\ \longrightarrow \mathcal{K}^i[T^{0\mu}] \end{matrix}$$

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BACKWARD DERIVATIVE

$$\nabla_i^- f(\boldsymbol{x}) = \frac{f(\boldsymbol{x}) - f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}})}{\delta x} \to \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x)$$

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Fluid dynamics in the **conservation form**

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space discretization

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NEUTRAL DERIVATIVE

$$\nabla_i^{(0)} f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\,\hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Fluid dynamics in the **conservation form**

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For a lattice of size L with N points per direction and lattice spacing $\delta x = L/N$ we have several possibilities

NEUTRAL DERIVATIVE

$$\nabla_i^{(0)} f(\boldsymbol{x}) = \frac{f(\boldsymbol{x} + \delta x\ \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\ \hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Simpler at higher orders if fields «live» at lattice sites

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

$\partial_0 T^{00} = -\,\partial_j T^{j0} \rightarrow \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{j0}$

$\partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \rightarrow \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{ji}$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x} + \delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\,\hat{\boldsymbol{\imath}})}{2\delta x} \rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^2)$$

Fluid dynamics often requires higher order spatial derivatives (shocks, nonlinearities...)

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{ji} \end{cases}$$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x} + \delta x\, \hat{\boldsymbol{\imath}}) - f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(4)} = \frac{-f(\boldsymbol{x} + 2\delta x\, \hat{\boldsymbol{\imath}}) + 8f(\boldsymbol{x} + \delta x\, \hat{\boldsymbol{\imath}}) - 8f(\boldsymbol{x} - \delta x\, \hat{\boldsymbol{\imath}}) + f(\boldsymbol{x} - 2\delta x\, \hat{\boldsymbol{\imath}})}{12\delta x}$$

$$\rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^4)$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

NEUTRAL DERIVATIVE

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(2)} = \frac{f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}})}{2\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(4)} = \frac{-f(\boldsymbol{x}+2\delta x\,\hat{\boldsymbol{\imath}}) + 8f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - 8f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}}) + f(\boldsymbol{x}-2\delta x\,\hat{\boldsymbol{\imath}})}{12\delta x}$$

$$\left[\nabla_i^{(0)} f(\boldsymbol{x})\right]^{(6)} = \frac{f(\boldsymbol{x}+3\delta x\,\hat{\boldsymbol{\imath}}) - 9f(\boldsymbol{x}+2\delta x\,\hat{\boldsymbol{\imath}}) + 45f(\boldsymbol{x}+\delta x\,\hat{\boldsymbol{\imath}}) - 45f(\boldsymbol{x}-\delta x\,\hat{\boldsymbol{\imath}}) + 9f(\boldsymbol{x}-2\delta x\,\hat{\boldsymbol{\imath}}) - f(\boldsymbol{x}-3\delta x\,\hat{\boldsymbol{\imath}})}{60\delta x}$$

$$\rightarrow \partial_i f(\boldsymbol{x})\Big|_{\boldsymbol{x}} + \mathcal{O}(\delta x^6)$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{j0} \\ \partial_0 T^{0i} = -\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{ji} \end{cases}$$

↑ Runge-Kutta order $(\Delta t)^N$

↑ Neutral derivative order $(\Delta t)^M$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

$\longrightarrow$ $\mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{j0}$

$\longrightarrow$ $\mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{ji}$

Runge-Kutta order $(\Delta t)^N$

Neutral derivative order $(\Delta t)^M$

A special property of the **CONSERVATION FORM**

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} T^{ji} \end{cases}$$

Runge-Kutta order $(\Delta t)^N$

Neutral derivative order $(\Delta t)^M$

A special property of the **CONSERVATION FORM**

When using periodic boundary conditions we have that (*Gauss theorem*)

$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0$$

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} & \longrightarrow \quad \mathcal{K}^0[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{j}0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] & \longrightarrow \quad \mathcal{K}^i[T^{0\mu}] \equiv \nabla_j^{(0)} \mathrm{T}^{\mathrm{ji}} \end{cases}$$

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Neutral derivative order $(\Delta t)^M$

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When using periodic boundary conditions we have that (*Gauss theorem*)

$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0 \longrightarrow \partial_0 \left[\sum_{all\ lattice\ points\ n} T^{0\mu}(n) \right] = 0$$

Fluid dynamics in the **conservation form**

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Runge-Kutta order $(\Delta t)^N$

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A special property of the **CONSERVATION FORM**

When using periodic boundary conditions we have that (*Gauss theorem*)

$$\sum_{all\ lattice\ points\ n} \nabla_j T^{j\mu}(n) = 0 \longrightarrow \partial_0 \langle T^{0\mu} \rangle = 0$$

Average $T^{0\mu}$ conserved at machine precision!

Fluid dynamics in the **conservation form**

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji} \end{cases}$$

An alternative form is obtained by substituting $T^{\mu\nu}$ with its expression in terms of the fluid primitive variables ρ and $\boldsymbol{u}$

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$

$$T^{00} = \rho(1 + c_s^2)\gamma^2 - c_s^2\, \rho$$

$$T^{0i} = \rho(1 + c_s^2)\gamma^2 u^i$$

$$T^{ji} = \rho(1 + c_s^2)\gamma^2 u^j u^i + c_s^2 \rho\, \delta^{ji}$$

Fluid dynamics in the **conservation form**

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$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j(\rho\,\gamma^2\,u^j)$$

$$T^{\mu\nu} = (\rho + p)\,u^\mu u^\nu + p\,\eta^{\mu\nu}$$

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Fluid dynamics in the **conservation form**

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$$\partial_0[\rho(1 + c_s^2)\gamma^2 - c_s^2 \rho] = -(1 + c_s^2)\partial_j(\rho\, \gamma^2\, u^j)$$

$$\partial_0\left[\rho(1 + c_s^2)\gamma^2 u^i\right] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^i u^j\right) - c_s^2 \partial_i \rho$$

Fluid dynamics in the **conservation form**

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An alternative form is obtained by substituting $T^{\mu\nu}$ with its expression in terms of the fluid primitive variables ρ and $\boldsymbol{u}$

$$\partial_0[\rho(1+c_s^2)\gamma^2 - c_s^2\rho] = -(1+c_s^2)\partial_j\left(\rho\,\gamma^2\,u^j\right) = \mathcal{K}^0$$

$$T^{\mu\nu} = (\rho + p)\,u^\mu u^\nu + p\,\eta^{\mu\nu}$$

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$$T^{\mu\nu} = (\rho + p)\,u^\mu u^\nu + p\,\eta^{\mu\nu}$$

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$$= \mathcal{K}^i$$

$$T^{00} = \rho(1+c_s^2)\gamma^2 - c_s^2\,\rho$$

Taking the scalar product of the latter equation with u^i and then subtracting from it the former we get

$$T^{0i} = \rho(1+c_s^2)\gamma^2 u^i$$

$$T^{ji} = \rho(1+c_s^2)\gamma^2 u^j u^i + c_s^2\rho\,\delta^{ji}$$

$$\partial_0\rho = \mathcal{K}^0 - u_i\mathcal{K}^i$$

Fluid dynamics in the **conservation form**

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$$\partial_0[\rho(1 + c_s^2)\gamma^2 - c_s^2 \rho] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^j\right) = \mathcal{K}^0$$

$$\partial_0\left[\rho(1 + c_s^2)\gamma^2 u^i\right] = -(1 + c_s^2)\partial_j\left(\rho\, \gamma^2\, u^i u^j\right) - c_s^2 \partial_i \rho$$

$$= \mathcal{K}^i$$

Taking the scalar product of the latter equation with u^i and then subtracting from it the former we get

$$\partial_0 \rho = \mathcal{K}^0 - u_i \mathcal{K}^i$$

$$\longrightarrow \rho(1 + c_s^2)\partial_0 \gamma^2 = \mathcal{K}^0 - [(1 + c_s^2)\gamma^2 - c_s^2][\mathcal{K}^0 - u_i \mathcal{K}^i]$$

Fluid dynamics in the **conservation form**

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Taking the scalar product of the latter equation with u^i and then subtracting from it the former we get

$$\partial_0 \rho = \mathcal{K}^0 - u_i \mathcal{K}^i$$

$$\rho(1 + c_s^2)\partial_0 \gamma^2 = \mathcal{K}^0 - [(1 + c_s^2)\gamma^2 - c_s^2][\mathcal{K}^0 - u_i \mathcal{K}^i]$$

Fluid dynamics in the **non-conservation form**

After a few manipulations we arrive at the **NON-CONSERVATION FORM** of fluid dynamics

$$\partial_0 \ln \rho = -\frac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

$$\partial_0 u_i = -\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right) u_i - \frac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i \frac{c_s^2}{(1-c_s^2 u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

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After discretizing the RHS we are left with a system of equations of the form

$$\partial_0 \ln\rho = \mathcal{G}^0[\ln\rho, u]$$

$$\partial_0 u_i = \mathcal{G}^i[\ln\rho, u]$$

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After discretizing the RHS we are left with a system of equations of the form

$$\partial_0 \ln\rho = \mathcal{G}^0[\ln\rho, u]$$ → The RHS depends on the fluid variables themselves

$$\partial_0 u_i = \mathcal{G}^i[\ln\rho, u]$$

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$$\partial_0 \ln\rho = \mathcal{G}^0[\ln\rho, u]$$ → The RHS depends on the fluid variables themselves

$$\partial_0 u_i = \mathcal{G}^i[\ln\rho, u]$$

Natural algorithm for timestepping → **explicit Runge-Kutta**

[See Lecture 3]

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$$\partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,u_i - \frac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i\frac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

Fluid dynamics in the **non-conservation form**

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \to \left[\nabla_\mathrm{j}^{(0)}\mathrm{u}_\mathrm{j}\right]^{(N)}$$

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\nabla\cdot\boldsymbol{u} \rightarrow \left[\nabla_\mathrm{j}^{(0)}\mathrm{u}_\mathrm{j}\right]^{(N)} \qquad\qquad (\boldsymbol{u}\cdot\nabla)\ln\rho \rightarrow u_\mathrm{j}\left[\nabla_\mathrm{j}^{(0)}\ln\rho\right]^{(N)}$$

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We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\nabla\cdot\boldsymbol{u} \to \left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_{\mathrm{j}}\right]^{(N)}$$

$$(\boldsymbol{u}\cdot\nabla)\ln\rho \to u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\ln\rho\right]^{(N)}$$

$$(\boldsymbol{u}\cdot\nabla)\,u_i \to u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_i\right]^{(N)}$$

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After a few manipulations we arrive at the **NON-CONSERVATION FORM** of fluid dynamics

$$\partial_0 \ln\rho = -\frac{1+c_s^2}{1-c_s^2u^2}\left[\nabla\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\nabla)\ln\rho\right]$$

$$\partial_0 u_i = -(\boldsymbol{u}\cdot\nabla)\,u_i - \frac{\mathrm{c}_s^2}{1+\mathrm{c}_s^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i\frac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\nabla\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\nabla)\ln\rho\right]$$

We can easily discretize the RHS at order $(\delta x)^N$ considering $\ln\rho$ and $\boldsymbol{u}$ living at lattice sites

$$\nabla\cdot\boldsymbol{u} \rightarrow \left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_{\mathrm{j}}\right]^{(N)} \qquad (\boldsymbol{u}\cdot\nabla)\ln\rho \rightarrow u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\ln\rho\right]^{(N)}$$

$$(\boldsymbol{u}\cdot\nabla)\,u_i \rightarrow u_{\mathrm{j}}\left[\nabla_{\mathrm{j}}^{(0)}\mathrm{u}_i\right]^{(N)} \qquad \nabla_i\ln\rho \rightarrow \left[\nabla_i^{(0)}\ln\rho\right]^{(N)}$$

CONSERVATION vs NON-CONSERVATION FORM

$$\begin{cases} \partial_0 T^{00} = -\,\partial_j T^{j0} \\ \partial_0 T^{0i} = -\,\partial_j T^{ji}[T^{0\mu}] \end{cases}$$

$$r^2 = \frac{T^{0i}T^{0i}}{(T^{00})^2} \qquad \gamma^2 = \frac{1}{2(1-r^2)}\left[1 - \frac{2r^2c_s^2}{1+c_s^2} + \sqrt{1 - \frac{4r^2c_s^2}{(1+c_s^2)^2}}\right]$$

$$T^{ji}[T^{0\mu}] = \frac{T^{0j}T^{0i}}{T^{00}}\left[1 - \frac{1}{\gamma^2}\,\frac{c_s^2}{1+c_s^2}\right] + \delta^{ji}\,T^{00}\,\frac{c_s^2}{\gamma^2(1+c_s^2) - c_s^2}$$

$$\begin{cases} \partial_0 \ln\rho = -\dfrac{1+c_s^2}{1-c_s^2u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \\ \partial_0 u_i = -\,(\boldsymbol{u}\cdot\boldsymbol{\nabla})\;u_i - \dfrac{c_s^2}{1+c_s^2}\dfrac{\nabla_i\ln\rho}{\gamma^2} + u_i\dfrac{c_s^2}{(1-c_s^2u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right] \end{cases}$$

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Both forms can be solved with a Runge-Kutta timestepping scheme and neutral derivatives

CONSERVATION vs NON-CONSERVATION FORM

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Which one is better?

CONSERVATION vs NON-CONSERVATION FORM

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Which one is better? It depends on the physical problem!

Beyond perfect fluid in flat spacetime

Our starting point was a perfect fluid $$T_{pf}^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} + p\, \eta^{\mu\nu}$$

$$\partial_{\mu} T_{pf}^{\mu\nu} = 0$$

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How to discretize viscous (non-perfect) fluids?

$$\partial_{\mu} T_{pf}^{\mu\nu} = f_{viscosity}^{\nu}$$

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How to discretize viscous (non-perfect) fluids?

How to deal with a coupled $U(1)$ gauge field?

$$\partial_\mu T^{\mu\nu}_{pf} = f^\nu_{viscosity} + f^\nu_{Lorentz}$$

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See Part II

$$\partial_\mu T^{\mu\nu}_{pf} = f^\nu_{viscosity} + f^\nu_{Lorentz} + f^\nu_{Hubble}$$